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**LINEAR
TRANSIENT ANALYSIS**

LINEAR TRANSIENT ANALYSIS

VOLUME II

Two-Terminal-Pair Networks
Transmission Lines

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NEW YORK · JOHN WILEY & SONS, INC.
LONDON · CHAPMAN & HALL, LIMITED

621.31921
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Library of Congress Catalog Card Number: 34-8791

PRINTED IN THE UNITED STATES OF AMERICA

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PREFACE

This second volume of *Linear Transient Analysis* covers essentially the second semester of the basic graduate course on transient analysis required of all graduate students in electrical engineering at the Polytechnic Institute of Brooklyn. It is devoted to transient phenomena in passive and active two-terminal-pair networks, in filters, and in transmission lines. As stressed in the preface to the first volume, it is in connection with these more advanced subjects that the Fourier and Laplace transform methods have demonstrated very significant advantages over the classical and Heaviside's operational methods. Consequently, the first chapter of the present volume contains a brief review of only the Fourier and Laplace transform methods in order to make the volume self-contained. Actually, the presentation follows here the mathematically perhaps more satisfying sequence from the dual Fourier integral relations to the Laplace transforms and stresses the spectrum aspects for ready reference in practical problems of signal transmission through networks of given frequency characteristics.

Chapter 2 introduces the concept and matrix description of the two-terminal-pair network for which the term fourpole is preferred as shorter and unambiguous if we use the equivalence of the two terms as definition rather than accept the possible broader meaning of a four-terminal network. Because of the great advantage in the systematic treatment of extended networks composed of fourpoles in cascade, as in nearly all practical communication systems, the matrix notation is used throughout, though more as a matter of convenience in notation than as a real application of matrix analysis. All the necessary relations of simple matrix algebra are given in Appendix 4 together with selected references for further study for those interested in broader applications. The very considerable generalization of solutions for fourpole problems made possible by matrix notation should readily prove its desirability. A brief section on response to frequency-modulated signals concludes this chapter.

Wave filters or passive fourpole lines are treated in Chapter 3 with the mathematical discussions needed to cover the extension of the Laplace transform method to difference equations of the particular kind arising here. A brief review of mechanical and thermal analogues is included because of the identical mathematical formulation.

The complexity of the general fourpole response and its broad aspect of limited frequency characteristics as either a low-pass or band-pass network invites idealization of the network characteristic as first introduced by K. Küpfmüller. Chapter 4 is devoted to a fairly extensive discussion of this application of the Fourier transform which so far has only been included sketchily in books on network analysis.

Chapter 5 gives a systematic exposition of active fourpoles, such as electron tubes and transistors as far as they operate in the small signal region and can thus be considered linear devices. Feedback control circuits and systems have not been included because a number of excellent books are available on this subject. The basic theory of feedback systems is, of course, covered in the section on feedback amplifiers and can readily be transcribed for feedback control systems if one makes the pertinent adjustments for the usual differences in notation and in nomenclature.

In Chapter 6, the physical phenomena on transmission systems with distributed parameters are discussed first in a qualitative way in order to stress the background and limits of validity of the conventional engineering concept of transmission-line theory which is particularly applicable to very low frequency systems. The concept of traveling waves is developed with care and solutions for lossless and distortionless lines are derived. The standing wave solution and its significance for the transient behavior of lines concludes the chapter.

Chapter 7 is devoted entirely to the ideal cable because of its considerable practical importance. The first, and simple, solution was given by W. Thomson (Lord Kelvin) when he analyzed the electrical characteristics of a transatlantic telegraph cable. New extensions of the inverse Laplace transform are developed as required and pertinent series expansions are discussed.

Finally, Chapter 8 presents approximations and the rigorous solution for the general transmission line. Because of the tremendous mathematical complexities encountered, only the simplest types of terminations are considered.

The Appendices give, as in the first volume, a list of symbols and brief reviews of matrices and functions of a complex variable so as to render the volume nearly self-sufficient. However, for details of the various methods of linear transient analysis and illustrations on simple lumped circuits, it will be necessary to consult the first volume.

Because of the more advanced mathematical level, this volume will be most suitable either for a second term of a graduate course on linear transient analysis or for a separate course on transients in filters and lines. Because of the immediate use of Fourier and Laplace trans-

forms, a knowledge of the elements of functions of a complex variable should be presupposed. As an aid in a review of this prerequisite, the essential relations are briefly summarized in Appendix 5.

Although the organization of the material can be considered fairly conventional, several new items are included which require the test of time. Perhaps this is fitting as a contribution to the Centennial Commemoration of Polytechnic Institute of Brooklyn in 1954-1955.

ERNST WEBER

Brooklyn, New York
May, 1956

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NOTES FOR THE READER

The symbols are tabulated in Appendix 1.

Equations are numbered consecutively in each section; references to equations in different sections carry the section number; thus (5.4) means equation (4) in section 5.

References to *Linear Transient Analysis*, Volume I, to which this book is the sequel, are simply marked: Vol. I, p. 20; or Vol. I, section 3.

Appendix 3 is a bibliography; only books are listed to which several individual or collective references are made in the text. Such references for example, Cherry,^{A4} p. 80, give the page and the author, the superscript indicating number 4 of section A of the bibliography.

To aid in the correct interpretation of standard mathematical terms, a glossary is given in Appendix 2.

Introduction

1. THE FOURIER AND LAPLACE TRANSFORMS

The Fourier and Laplace transform methods are basically transformations of the linear integro-differential equations describing linear networks and systems into algebraic relationships for the transforms of the physical quantities which are to be evaluated. The first volume dealt with lumped-parameter networks and therefore involved only linear differential equations with constant coefficients, permitting a very simple demonstration of the inversion of the Laplace transform solution in Vol. I, p. 203.

We shall briefly review here both the Fourier and Laplace transform concepts and discuss the inversion process from the broader aspect of the applications to wave filter and transmission-line problems. In order not to separate completely the mathematical discussion from the methods of application, this chapter will be only a general exposition of the Fourier and Laplace transform methods, leaving many of the mathematical details for the main text, wherever they might be illustrated most appropriately.

1.1 The Fourier Integral

It is probably no exaggeration to introduce the Fourier integral as the key to the modern theory of linear differential equations; indeed, it has been the most reliable basis of the transformation theory. The essence of the Fourier integral theory is the representation of a large group of functions in terms of infinite integrals as follows: If a function $f(t)$ is integrable in every finite interval, if the integral of its absolute value,

$$\int_{-\infty}^{+\infty} |f(t)| dt$$

converges, and if $f(t)$ is of bounded variation in the neighborhood of t , then we can write the Fourier integral theorem in complex form

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} e^{j\omega(t-u)} f(u) du \quad (1)$$

If $f(t)$ has a finite discontinuity at point t , then the double integration gives the average

$$\frac{1}{2}[f(t^-) + f(t^+)]$$

of the values at either side of the discontinuity. The integration with respect to ω has to be taken in the Cauchy sense as principal value, i.e., as the limit

$$\lim_{M \rightarrow \infty} \int_{-M}^{+M} H(\omega) d\omega$$

The proof of the representation (1) is rather involved and generally employs the Dirichlet-type integrals.* The conditions stated above are sufficient, and generally necessary with the conventional kind of integration process.

We can separate the integral representation (1) into a tandem arrangement of the two integrals

$$\bar{F}(\omega) = \int_{-\infty}^{+\infty} f(u) e^{-j\omega u} du \quad (2)$$

and

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \bar{F}(\omega) e^{j\omega t} d\omega \quad (3)$$

which can be considered as a *dual pair*, the *Fourier transform pair* in complex form. $\bar{F}(\omega)$, the Fourier transform of $f(t)$, is obviously a complex number, which we might write

$$\bar{F}(\omega) = F_1(\omega) + jF_2(\omega) \quad (4)$$

From (2) it is also clear, since $f(u)$ had been assumed real, that by using the trigonometric equivalent of the complex exponential we can identify

$$F_1(\omega) = \int_{-\infty}^{+\infty} f(u) \cos \omega u du, \quad F_2(\omega) = - \int_{-\infty}^{+\infty} f(u) \sin \omega u du \quad (5)$$

Because of the fact that only $\cos \omega u$ contains ω and, indeed, is an even function of ω , $F_1(\omega)$ will also be an even function of ω ; correspondingly, $F_2(\omega)$ will be an odd function of ω .

* See particularly H. S. Carslaw, *Introduction to the Theory of Fourier's Series and Integrals*, 3rd edition, 1930, pp. 310-312; reprinted by Dover Publications, New York, 1946. Also Bochner,^{D1} pp. 39-40; Titchmarsh,^{D19} pp. 402-408.

Alternatively, we can write the Fourier transform in polar notation, namely

$$\bar{F}(\omega) = |\bar{F}(\omega)|e^{j\phi(\omega)} \quad (6)$$

with

$$|\bar{F}(\omega)| = F(\omega) = [F_1^2(\omega) + F_2^2(\omega)]^{1/2}$$

$$\phi(\omega) = \tan^{-1} \frac{F_2(\omega)}{F_1(\omega)} \quad (7)$$

where $F(\omega)$ is the *amplitude function*, necessarily an even function of ω because it contains the squares of F_1 and F_2 ; and where $\phi(\omega)$ is the *phase function*, necessarily an odd function of ω .

Convergence of Fourier integrals. Examining the integral (2) which defines the Fourier transform, we see readily the reason for the requirements of integrability of $f(t)$ and, in particular, of the existence of the infinite integral of the absolute value of $f(t)$, because the absolute value of the exponential factor $\exp(-j\omega t)$ is unity. For practical applications, this could be a severe limitation. In fact, the unit step, defined by

$$1 = H(t) = S_{-1}(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases} \quad (8)$$

with 1 used by Heaviside, $H(t)$ indicating "Heaviside's function," and $S_{-1}(t)$ employed by Campbell-Foster,^{E1} has no Fourier transform in the strict sense of the definition given above. The same is true for the trigonometric functions $\sin \Omega t$, and $\cos \Omega t$. To make the Fourier integral method useful in these cases of extreme practical importance, we might introduce a convergence factor e^{-ct} and actually use the modified function

$$g(t) = e^{-ct}f(t) \quad (9)$$

This leads to the Fourier transform

$$\bar{G}(\omega) = \int_{-\infty}^{+\infty} f(u)e^{-(j\omega+c)u} du = \bar{F}(\omega - jc) \quad (10)$$

If then the limit exists for $c \rightarrow 0$, so that $\bar{G}(\omega) = \bar{F}(\omega - jc) \rightarrow \bar{F}(\omega)$, we call $\bar{F}(\omega)$ the Fourier transform of $f(t)$ in the Abel-Poisson sense; see Doetsch,^{D5} p. 47, as pointed out in Vol. I, p. 281.

The integral (3) is usually called the inverse Fourier transform to $\bar{F}(\omega)$ and necessarily leads to a real function of the parametric variable t . Obviously, if $\bar{F}(\omega)$ is known to be a Fourier transform, then $f(t)$ exists and the integral can be evaluated. We need only to separate $\bar{F}(\omega)$ into real and imaginary parts as in (4) and apply to each separately the same requirements as applied to $f(t)$ in the integral (2).

But, utilizing the fact that $F_1(\omega)$ and $F_2(\omega)$ are even and odd, respectively, in ω , we realize that in the infinite integrals $F_1(\omega) \sin \omega t$ and $F_2(\omega) \cos \omega t$ do not give any contribution because they are odd functions of ω . This means, however, that the whole imaginary part vanishes, as indeed it must, so that we are left with the result

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (F_1(\omega) \cos \omega t - F_2(\omega) \sin \omega t) d\omega \quad (11)$$

It would be desirable, of course, upon meeting a function $\bar{F}(\omega)$ to ascertain at once whether or not it is a possible Fourier transform and thus apply at once (11). Apparently, such general mathematical classification has not yet been feasible. Fortunately, in all our applications we shall deal with physical problems for which no doubt exists as to the nature of $\bar{F}(\omega)$. In the few instances where nonrealizable network characteristics are assumed we shall point out this fact explicitly.

Real forms of Fourier integrals. We started with the complex form of the Fourier integral theorem (1) which is actually due to Cauchy, who also introduced the concept of transformation; see Bochner,^{D1} p. 220, notes 2 and 6. The real form is demonstrated in Vol. I, p. 270, and is conventionally written

$$f(t) = \int_0^{\infty} [\Phi(\omega) \cos \omega t + \psi(\omega) \sin \omega t] d\omega \quad (12)$$

where

$$\begin{aligned} \Phi(\omega) &= \frac{1}{\pi} \int_{-\infty}^{+\infty} f(u) \cos \omega u du \\ \psi(\omega) &= \frac{1}{\pi} \int_{-\infty}^{+\infty} f(u) \sin \omega u du \end{aligned} \quad (13)$$

are the real Fourier coefficient functions. Comparing (13) with (5), we find at once

$$\pi\Phi(\omega) = F_1(\omega), \quad \pi\psi(\omega) = -F_2(\omega) \quad (14)$$

which also checks as we compare (12) with (11), keeping in mind that the integrand is an even function of ω , so that the limits can be reduced to 0 to ∞ with the factor $\frac{1}{2}$ omitted. The shift of the factor $1/\pi$ from the time function to the transforms is arbitrary but conventional. We note also that $\Phi(\omega)$ must be an even, $\psi(\omega)$ an odd function of ω which was pointed out in Vol. I, p. 276. At any finite discontinuity, we need

to substitute for $f(t)$ on the left-hand side in (12) the average value at either side of the discontinuity as discussed in connection with (1).

For practical applications the most useful collection of Fourier integrals is given in Campbell-Foster,^{E1} where a slightly different notation is used from above. Observing that ω enters into the integral (2) only in combination with j , we might introduce the parameter

$$p = j\omega = j2\pi f \quad (15)$$

and write instead of $\bar{F}(\omega)$ the symbol $F(p)$ which has the identical meaning but appears with real coefficients, since the imaginary unit is absorbed in p . We can thus write for (2)

$$F(p) = \int_{-\infty}^{+\infty} G(t)e^{-pt} dt \quad (16)$$

where we also substituted $G(t)$ for the time function $f(t)$ in order to arrive exactly at the Campbell-Foster notation, except that we still have t as the time variable, whereas g is used there. The inversion becomes

$$G(t) = \int_{-\infty}^{+\infty} F(p)e^{pt} df \quad (17)$$

where the last form utilizes $\omega = 2\pi f$, keeping it a real integral along the frequency axis.

As example we might take the function shown in Fig. 1.1

$$G(t) = \alpha te^{-\alpha|t|} \quad (18)$$

which is an odd function and definitely integrable in the infinite limits. We obtain readily by integration by parts

$$\int_0^{+\infty} te^{-\beta t} dt = \left. \frac{t}{-\beta} e^{-\beta t} \right|_0^{\infty} - \int_0^{\infty} \frac{e^{-\beta t}}{-\beta} dt = + \frac{1}{\beta^2} \quad (19)$$

We apply this to (16) with (18) for $G(t)$, breaking the interval of integration so as to permit elimination of the absolute signs,

$$\begin{aligned} F(p) &= \int_{-\infty}^0 \alpha te^{+\alpha t} e^{-pt} dt + \int_0^{\infty} \alpha te^{-\alpha t} e^{-pt} dt \\ &= \alpha \left(\frac{-1}{(p - \alpha)^2} + \frac{1}{(p + \alpha)^2} \right) \end{aligned}$$

This result can be verified in Campbell-Foster,^{E1} pair 442. By contraction we obtain simpler

$$F(p) = - \frac{4p\alpha^2}{(p^2 - \alpha^2)^2} \quad (20)$$

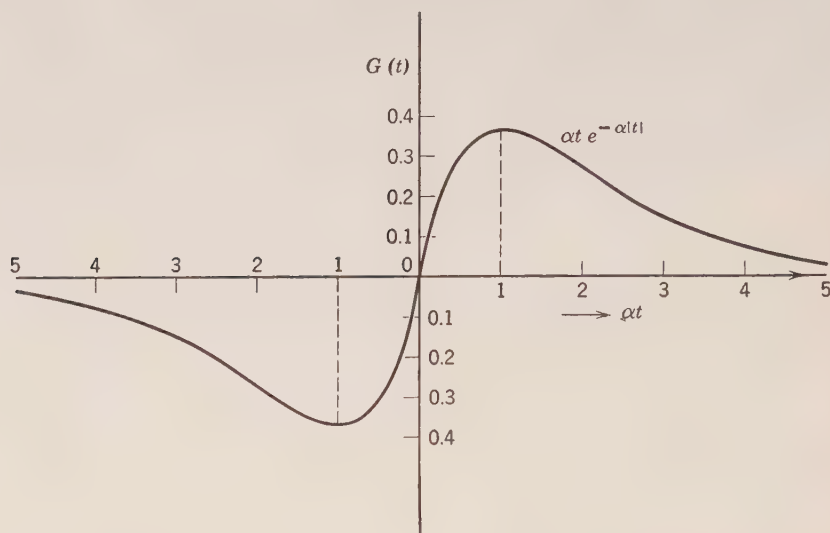


Fig. 1.1. Example of a time function for Fourier integral representation.

TABLE 1.1
SHORT TABLE OF FOURIER TRANSFORM PAIRS

No.	Time Domain, Validity $f(t)$	Transform Domain $F(p)$ or $F(\omega)$
1	$e^{-\alpha t}$ $t > 0$	$\frac{1}{p + \alpha}$ $\frac{1}{j\omega + \alpha}$
2	$-e^{\alpha t}$ $t < 0$	$\frac{1}{p - \alpha}$ $\frac{1}{j\omega - \alpha}$
3	$-\frac{1}{2\alpha} e^{-\alpha t } = -\frac{1}{2\alpha} \begin{cases} e^{-\alpha t} & t > 0 \\ e^{\alpha t} & t < 0 \end{cases}$	$\frac{1}{p^2 + \alpha^2}$ $\frac{-1}{\omega^2 + \alpha^2}$
4	$-\frac{1}{\alpha + \beta} \begin{cases} e^{-\alpha t} & t > 0 \\ e^{\beta t} & t < 0 \end{cases}$	$\frac{1}{(p + \alpha)(p - \beta)}$ $\frac{1}{(j\omega + \alpha)(j\omega - \beta)}$
Simple Pulses:		
11	1 $ t < \tau$	$\frac{2}{p} \sinh(\tau p)$ $2\tau \frac{\sin \omega \tau}{\omega \tau}$
12	t $ t < \tau$	$\frac{2}{p^2} [\sinh(\tau p) - \tau p \cosh(\tau p)]$ $\frac{2\tau}{j\omega} \left(\frac{\sin \omega \tau}{\omega \tau} - \cos \omega \tau \right)$
13	$e^{j\omega_0 t}$ $ t < \tau$	$\frac{2}{p - j\omega_0} \sinh[\tau(p - j\omega_0)]$ $2\tau \frac{\sin(\omega - \omega_0)\tau}{(\omega - \omega_0)\tau}$
14	$\begin{cases} -1/2 & 0 < t < \tau \\ +1/2 & -\tau < t < 0 \end{cases}$	$\frac{1}{p} [\cosh(\tau p) - 1]$ $\frac{1}{j\omega} (\cos \omega \tau - 1)$
Bilateral Continuous Functions:		
21	$\frac{\tau}{\pi(t^2 + \tau^2)}$ all t	$e^{-\tau p }$ $e^{-\tau \omega }$ (see pair 3)
22	$\frac{\tau^2 - t^2}{\pi(\tau^2 + t^2)^2}$ all t	$ p e^{-\tau p }$ $ \omega e^{-\tau \omega }$
23	$\frac{1}{2(\pi a)^{1/2}} e^{-\frac{t^2}{4a}}$ all t	e^{-ap^2} $e^{-a\omega^2}$
24	$\frac{1}{2(\pi a)^{1/2}} \sin\left(\frac{t^2}{4a} - \frac{\pi}{4}\right)$ all t	$\sin(ap^2)$ $-\sin(a\omega^2)$
25	$\frac{1}{2(\pi a)^{1/2}} \cos\left(\frac{t^2}{4a} - \frac{\pi}{4}\right)$ all t	$\cos(ap^2)$ $\cos(a\omega^2)$

If we like, we can reintroduce here $p = j\omega$ and separate real and imaginary parts. Thus,

$$\bar{F}(\omega) = -j \frac{4\omega\alpha^2}{(\omega^2 + \alpha^2)^2} = jF_2(\omega)$$

is a purely imaginary value, since the transform is entirely an odd function of ω . In this case, we can easily apply (11) and find the Fourier integral representation of (18) as

$$G(t) = \frac{1}{2\pi} 4\alpha^2 \int_{-\infty}^{+\infty} \frac{\omega}{(\omega^2 + \alpha^2)^2} \sin \omega t d\omega \quad (21)$$

which is an interesting result and not entirely expected. Table 1.1, which is a reproduction of Table 6.1 from Vol. I., p. 285, lists a few pairs of Fourier integrals of similar types; see particularly pair 3.

1.2 The Fourier Spectrum Functions

The representation of functions by means of the Fourier integral duality can also be interpreted as a change from the domain of one variable to the domain of a different variable. If we like to stress this viewpoint of “*transformation*,” then we can consider on the one hand the real function $G(t)$ of the real variable t , or of any other convenient real variable, and on the other hand the complex function $\bar{F}(\omega)$ of the real variable ω , or the function pair $\Phi(\omega)$ and $\psi(\omega)$, as mutually associated, as mutual “*images*,” or as transforms into related domains. The important aspect is the correspondence of a single real time function to a pair of real frequency functions usually expressed in terms of the angular frequency ω , and vice versa. It is customary to refer to the ω -functions as *spectrum functions*.

The concept of the spectrum function was introduced in Vol. I, chapter 6, in connection with the Fourier series which lead to an equidistant set of *harmonic spectral lines*. The spectrum concept for Fourier integrals leads to continuous spectra of infinite extent but of infinitesimal amplitude at any one frequency. Strictly speaking, one should only use integrals of the spectrum functions over small but finite frequency intervals or frequency bands in order to be able to assign finite values. However, it has become customary to plot directly either $F_1(\omega)$ and $F_2(\omega)$ from (4) and (5) as the “in-phase” and “quadrature” spectrum functions; or $F(\omega)$ and $\phi(\omega)$ from (7) as the more conventional amplitude and phase functions. To establish a better understanding of these basic concepts and tools of analysis and

synthesis, it might be best to carry through a few illustrative examples with appropriate discussions of the basic aspects.

Abrupt pulse form spectra. Suppose $G(t)$ is a signal of given time variation, say a single pulse of the exponential type

$$G(t) = \begin{cases} Ve^{-\alpha t} & \text{for } 0 < t < T \\ 0 & \text{for } t > T \end{cases} \quad (22)$$

where $\alpha T = m$ might be taken as a describing parameter as indicated in Fig. 1.2. The rectangular pulse is the special case $m = 0$. We find

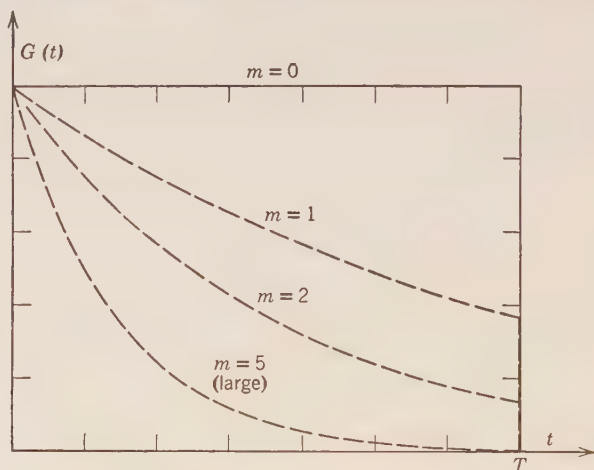


Fig. 1.2. Single pulses of exponential type $e^{-\alpha t}$; $m = \alpha T = 0$ is rectangular pulse.

readily the complex Fourier transform by (2)

$$\bar{F}(\omega) = V \int_0^T e^{-\alpha u} e^{-j\omega u} du = V \frac{1 - e^{-(\alpha + j\omega)T}}{\alpha + j\omega} \quad (23)$$

This result can be interpreted as the addition of two Fourier transforms, the first

$$\frac{V}{\alpha + j\omega}$$

being simply the transform of the infinitely continuing exponential function $e^{-\alpha t}$. The second part is

$$- \frac{Ve^{-\alpha T}}{\alpha + j\omega} e^{-j\omega T}$$

the transform of an exponential function of amplitude $Ve^{-\alpha T}$ starting negatively at time $t = T$ as indicated by the delay factor $e^{-j\omega T}$.

Beyond $t = T$, the remaining part of the original exponential function and this new exponential function exactly cancel, which produces the zero value of the pulse. It is important to appreciate this superposition because it can simplify the evaluation of transforms of fairly complex pulse shapes. We could thus write instead of the expressions in (22)

$$G(t) = Ve^{-\alpha t} - (Ve^{-\alpha T})e^{-\alpha(t-T)}S_{-1}(t - T)$$

where the second part is the delayed exponential, $S_{-1}(t - T)$ being the unit step with inception at $t = T$.

We might now introduce $\omega T = x$ as a dimensionless variable and $\alpha T = m$ as the dimensionless characteristic parameter and obtain for (23)

$$\bar{F}(\omega) = VT \frac{1 - e^{-(m+jx)}}{m + jx} \quad (24)$$

Upon rationalization we find for the real and imaginary parts, respectively,

$$\begin{aligned} F_1(\omega) &= VT \frac{m(1 - e^{-m} \cos x) + xe^{-m} \sin x}{m^2 + x^2} \\ F_2(\omega) &= VT \frac{me^{-m} \sin x - x(1 - e^{-m} \cos x)}{m^2 + x^2} \end{aligned} \quad (25)$$

We see that $F_1(\omega)$ is an even and $F_2(\omega)$ is an odd function of x and therefore of ω , as expected. To visualize the dependence upon $x = \omega T$ and $m = \alpha T$, Fig. 1.3 shows the complex Fourier transform in graphical representation with x as the parameter along each individual curve for which m is a constant value. In this manner, Fig. 1.3 gives a unique representation in the complex plane of the respective pulse shapes of Fig. 1.2 which are functions of time. Writing this

$$G(t) \leftrightarrow \bar{F}(\omega)$$

we can conceive of this as a unique transformation in either direction.

We observe that the pulses $m = 0, 1, 2$ in Fig. 1.2 differ in values over the interval $0 < t < T$, all of them being zero thereafter. Similarly, the complex transforms are significantly different only over the interval $0 < \omega T < 2\pi$ of the angular frequency variable ω , or in other words, the significant spectrum contribution lies in the frequency band

$$0 < f < 1/T$$

This becomes even more impressive if we plot the two spectrum functions $F_1(\omega)$ and $F_2(\omega)$ against $x = \omega T$ as variable as is done in Fig. 1.4.

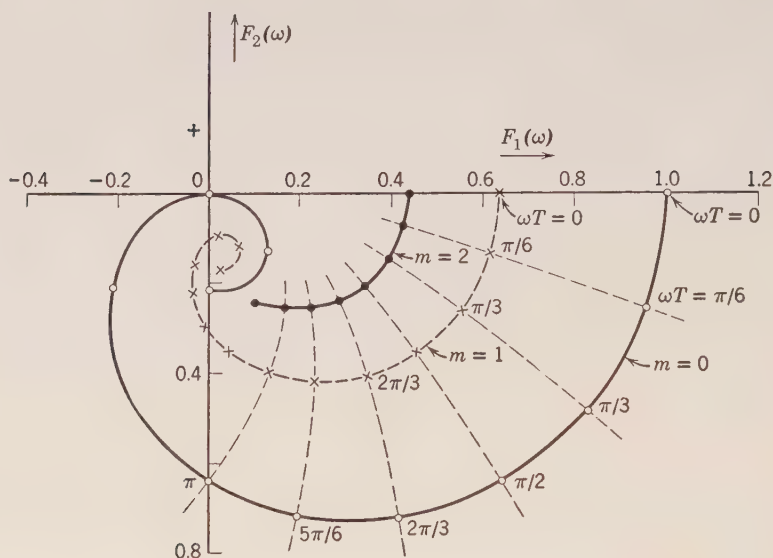


Fig. 1.3. The complex Fourier transforms $\bar{F}(\omega)$ of the pulse types shown in Fig. 1.1. Values in fractions of π are the values of $x = \omega T$ as the parameter.

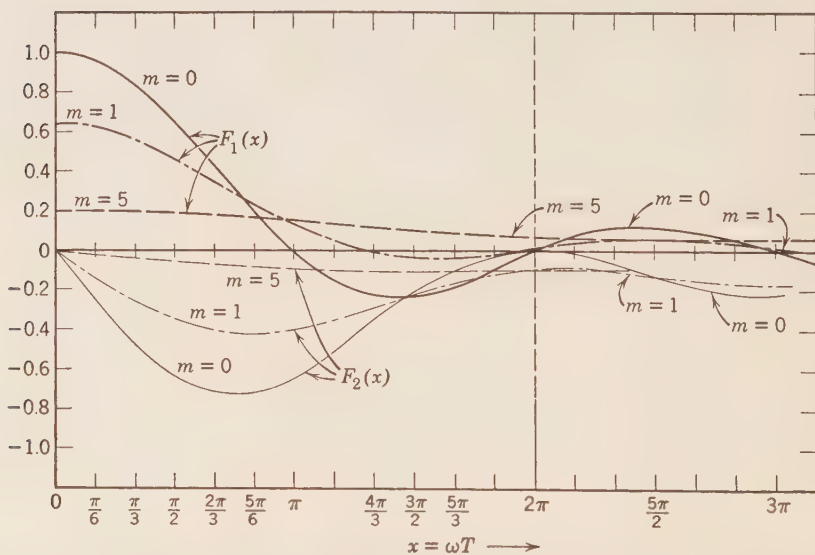


Fig. 1.4. Spectrum functions $F_1(\omega)$ and $F_2(\omega)$ for the pulses $m = 0, 1, 5$ in Fig. 1.2.

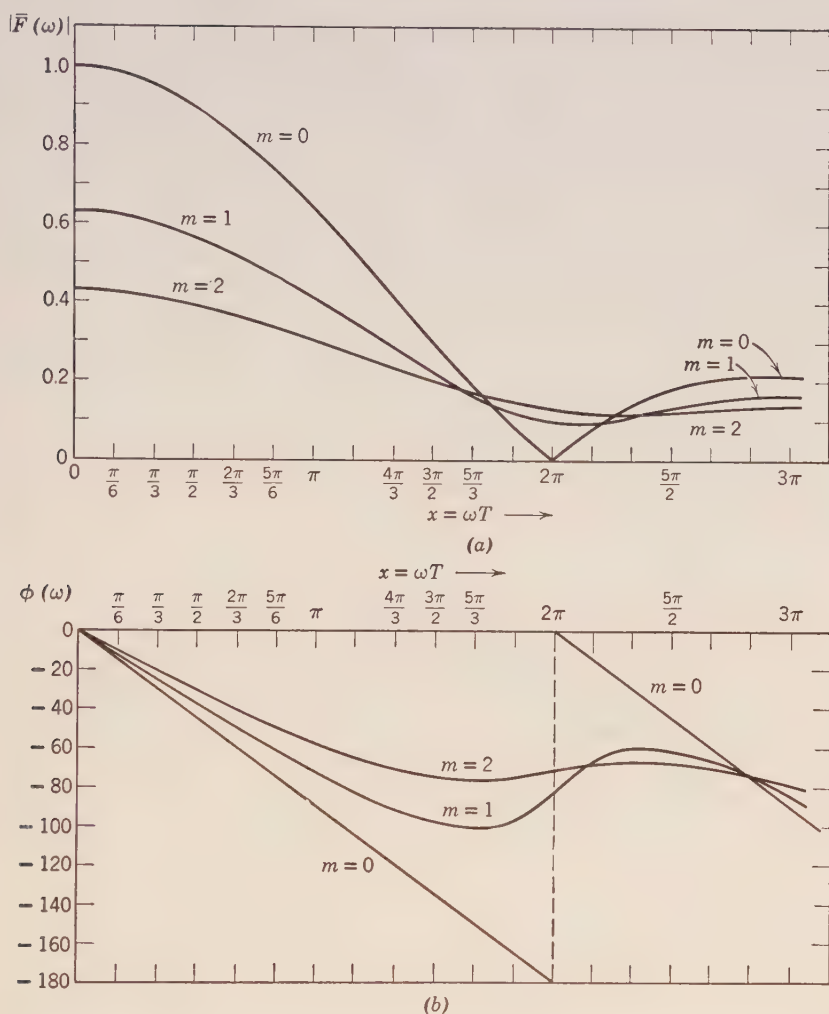


Fig. 1.5. Amplitude function (a) and phase function (b) of the pulse types $m = 0, 1, 2$ in Fig. 1.2.

Of course, Figs. 1.3 and 1.4 present the same type of information, Fig. 1.3 giving the "locus" of the complex function in terms of the variable ωT , Fig. 1.4 giving the more conventional presentation for real functions.

The pulse shape $m = 5$ cannot be considered directly with the others, because the whole phenomenon is essentially over at time T ; the pulse "duration" has become short compared with the interval of observation. In this case, it is necessary to define the pulse duration fairly

arbitrary, e.g., by selecting the time at which the value of the pulse has decreased to 5% of its initial value which is also its peak value. This would give the duration $\tau = 0.45T$, for which we would expect a necessary frequency band $0 < f < 2.22/T$ or a frequency band 2.22 times wider than for the previously considered pulses. The shorter the pulse duration, the wider must be the frequency band to give the significant spectrum contribution. This is quite obvious in Fig. 1.4.

Finally we might also plot the amplitude function $F(\omega)$ and the phase function $\phi(\omega)$ defined by (7). Figure 1.5 gives these for the pulse

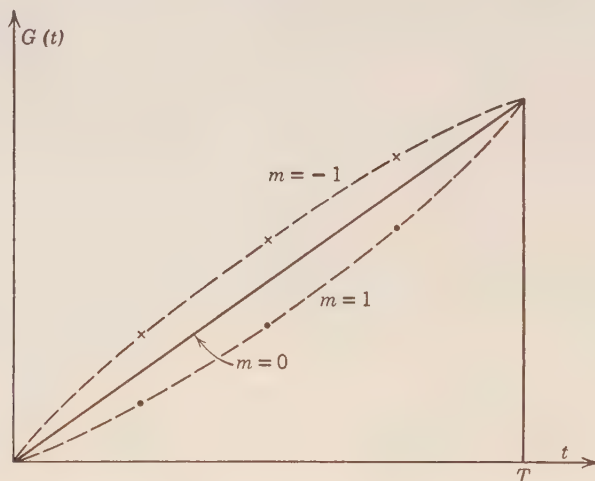


Fig. 1.6. Single pulses of sawtooth type, $m = 0$ being the conventional linear sawtooth.

shapes $m = 0, 1, 2$, and bears out the strong concentration of amplitude values within the range $0 < \omega T < 2\pi$. We should also notice the marked differences in the phase functions for the three pulses, emphasizing the importance of phase characteristics of transmitting networks.

Gradual pulse rise spectra. The pulses in Fig. 1.2 started abruptly with their peak values, taking the rectangular pulse as the limit. What about different types of pulses? Suppose we consider the pulse shapes in Fig. 1.6 which start at zero value, thus provide continuous transition at the origins, rise smoothly, and then terminate abruptly. Specifically the three pulse types are defined by

$$G(t) = \begin{cases} V \frac{e^{\alpha t} - 1}{e^{\alpha T} - 1}, & V \frac{t}{T}, & V \frac{1 - e^{-\alpha t}}{1 - e^{-\alpha T}} & t < T \\ 0, & 0, & 0, & t > T \end{cases} \quad (26)$$

whereby the center expression gives the linear pulse, and the first and third expressions the pulses rising initially less than and more than linearly, respectively. We can actually cover all three pulses by the single expression

$$G(t) = \begin{cases} V \frac{e^{m\tau} - 1}{e^m - 1} & \text{for } 0 < t < T \\ 0 & \text{for } t > T \end{cases} \quad (27)$$

where $m = \alpha T$ has values $m = 0$ for the linear pulse, and $m > 0$ for the first, $m < 0$ for the last pulse type in (26). $\tau = t/T$ is then the relative time. We chose specifically $m = +1$, and $m = -1$ to see the effect of comparatively small changes in the time function.

The complex Fourier transform can again be evaluated by (2). Taking the general form (27), we get with the further abbreviation $\omega T = x$,

$$\begin{aligned} \bar{F}(\omega) &= \frac{VT}{e^m - 1} \int_0^1 (e^{m\tau} - 1) e^{-jx\tau} d\tau \\ &= \frac{VT}{e^m - 1} \left(\frac{1}{m - jx} (e^{m-jx} - 1) + \frac{1}{jx} (e^{-jx} - 1) \right) \end{aligned}$$

Upon rationalization we find for the real and imaginary parts, respectively,

$$\begin{aligned} F_1(\omega) &= \frac{VT}{e^m - 1} \left(\frac{m(e^m \cos x - 1) + xe^m \sin x}{m^2 + x^2} - \frac{\sin x}{x} \right) \\ F_2(\omega) &= \frac{VT}{e^m - 1} \left(\frac{x(e^m \cos x - 1) - me^m \sin x}{m^2 + x^2} + \frac{1 - \cos x}{x} \right) \end{aligned} \quad (28)$$

It is again evident that $F_1(\omega)$ is an even and $F_2(\omega)$ an odd function of x and therefore of ω . Figure 1.7 shows the locus of the complex Fourier transform in terms of these quadrature components with x as the parameter and m as a fixed value. We observe the close similarity of the curves corresponding to the temporal pulse shapes in a 1-to-1 relationship which can be conceived as transformation from the $f(t)$ -plane to the $\bar{F}(\omega)$ -plane.

Even more so than in the previously discussed pulse shapes, all the significant difference between the complex Fourier transforms occurs in the interval $0 < x < 2\pi$, or the frequency band

$$0 < f < 1/T$$

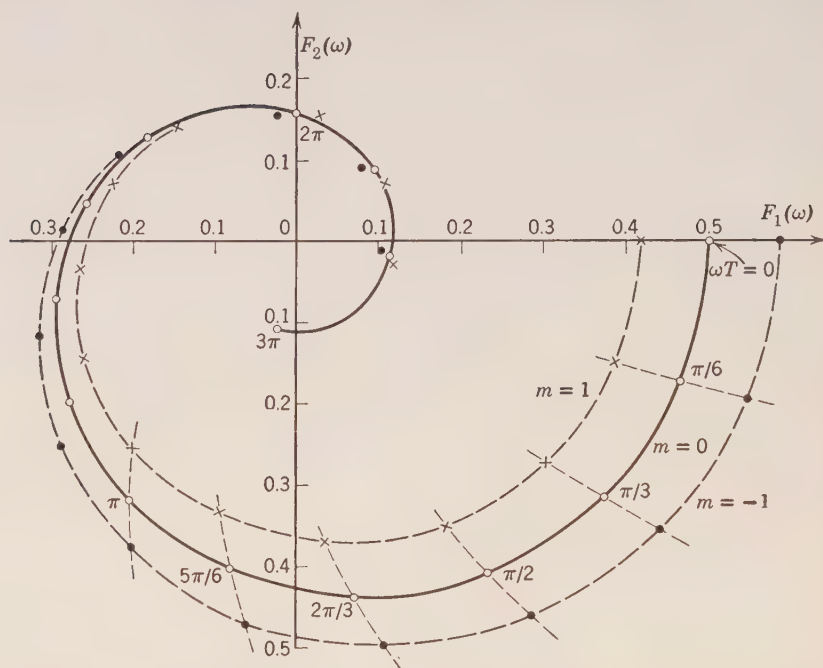


Fig. 1.7. The complex Fourier transform $\bar{F}(\omega)$ of the sawtooth pulses shown in Fig. 1.6. Values in fractions of π are the values of $x = \omega T$ as the parameter.

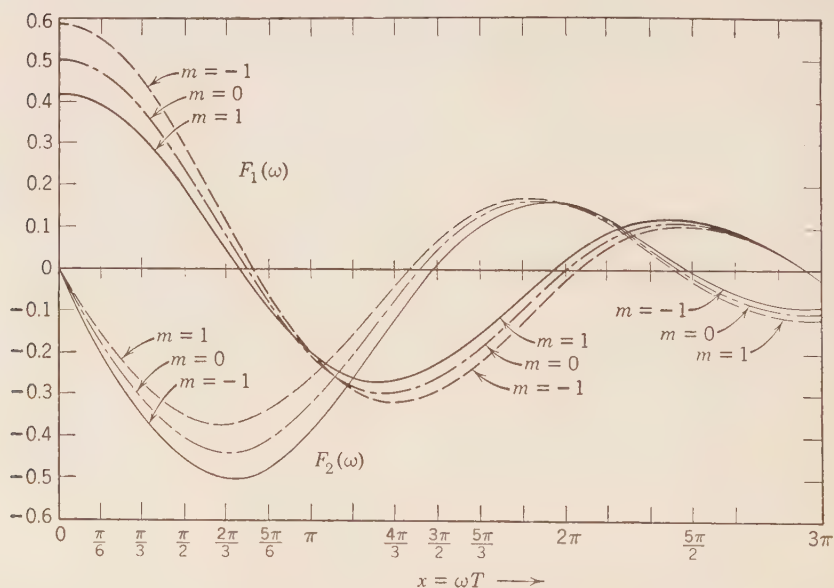


Fig. 1.8. Spectrum functions $F_1(\omega)$ and $F_2(\omega)$ for the sawtooth pulses in Fig. 1.6.

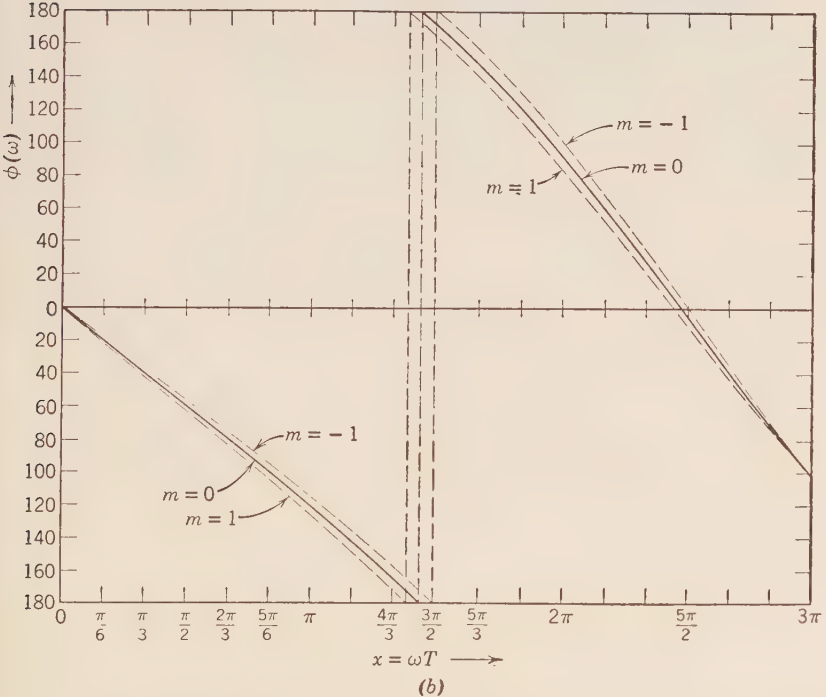
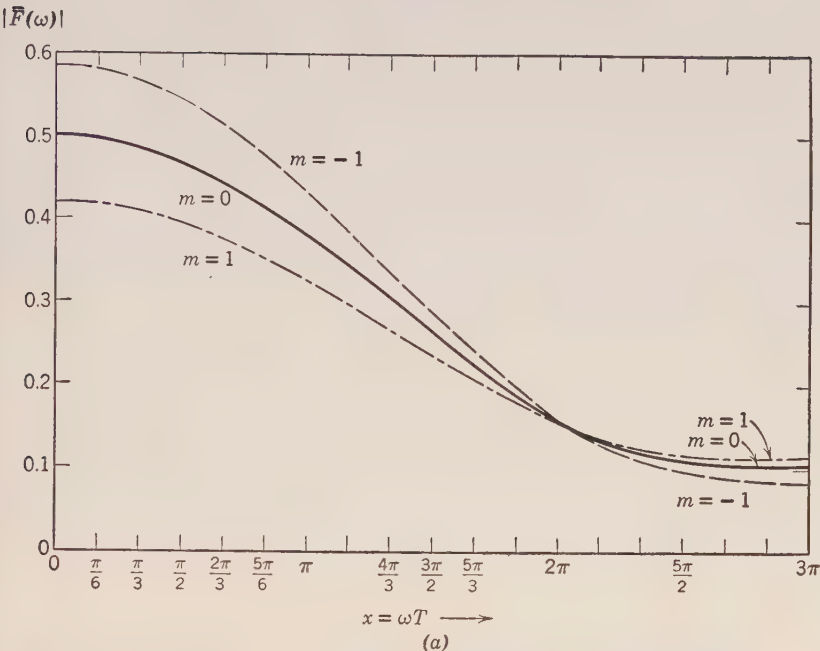


Fig. 1.9. Amplitude function (a) and phase function (b) of the sawtooth pulses in Fig. 1.6.

and as the pulse duration decreases the necessary frequency band increases, keeping the product constant at unity. We could therefore take the simple spectrum function of the linear sawtooth for all the pulse shapes beyond $x = 2\pi$.

Comparing now the two sets of complex Fourier transform loci in Figs. 1.3 and 1.7 we also note a considerable difference. The pulses with sharp leading edge have transforms lying entirely in the lower half plane of $\bar{F}(\omega)$, whereas the pulses with sharp trailing edge spiral widely around the origin. This might be of general importance and will be discussed again in chapter 4.

Plotting the two spectrum functions $F_1(\omega)$ and $F_2(\omega)$ against ω as in Fig. 1.8, we again note first the general similarity of these functions for the three different sawtooth pulse shapes of Fig. 1.6, and second the marked difference in appearance between this pulse group and the one whose spectrum functions are pictured in Fig. 1.4. The same comparisons are quite obvious for the amplitude and phase functions plotted for the sawtooth pulse group in Fig. 1.9 and for the exponential group in Fig. 1.5. It is particularly instructive to study the differences in the phase functions.

1.3 The Fourier Transform Method

The Fourier integral representation of arbitrary nonperiodic functions which satisfy the Dirichlet conditions (outlined in section 1.1) is of greatest value in the solution of linear differential equations in one or more variables. In particular it permits the transformation of the ordinary integro-differential equations of lumped-parameter networks into linear algebraic relations between the Fourier transforms of the unknowns which can be solved readily by conventional algebraic methods. The solutions for the transforms can then be converted back into the time functions describing the transient response of the network. Though the method has been discussed in some detail in Vol. I, chapter 6, we shall review briefly the main aspects of the use of Fourier transforms.

Application of Fourier transforms to a simple circuit. Let us assume as a typical ordinary linear integro-differential equation that of the series double-energy circuit

$$L \frac{di}{dt} + Ri + \frac{1}{C} \int i dt = v(t) \quad (29)$$

in which $i(t)$ is the unknown current as a function of time, $v(t)$ the active voltage as a disturbance function of the system, and R, L, C are the

characteristic constant parameters of the linear system. We perform a Fourier transformation of this integro-differential equation by multiplying each term by $e^{-j\omega t}$ and integrating in the infinite limits as indicated in (2). This gives on the right-hand side, since $v(t)$ must be specified,

$$\bar{V}(\omega) = \int_{-\infty}^{+\infty} v(t)e^{-j\omega t} dt = \mathfrak{F}v(t) \quad (30)$$

where the last form is simply the symbolic representation of the direct Fourier integral. If the function $v(t)$ is analytic for all values of t and if the indefinite integral exists, perhaps we call it $\bar{J}(\omega, t)$, then

$$\bar{V}(\omega) = \lim_{t \rightarrow \infty} [\bar{J}(\omega, +t) - \bar{J}(\omega, -t)]$$

This limit might be reached by means of convergence factors as discussed above in (9) and (10). In most transient problems the disturbance $v(t)$ can be assumed to start at $t = 0$, so that $v(t) = 0$ for $t < 0$. In this case, then, (30) has as lower limit $t = 0^+$, and

$$\bar{V}(\omega) = \lim_{t \rightarrow \infty} \bar{J}(\omega, t) - \lim_{t \rightarrow 0^+} \bar{J}(\omega, t)$$

where the stipulation 0^+ allows for any discontinuity of the function at $t = 0$. If we take

$$v(t) = \begin{cases} Ve^{-\alpha t} & t > 0 \\ 0 & t < 0 \end{cases} \quad (31)$$

then we get the indefinite integral

$$\bar{J}(\omega, t) = V \int e^{-(\alpha + j\omega)t} dt = V \frac{e^{-(\alpha + j\omega)t}}{-(\alpha + j\omega)}$$

and thus

$$\bar{V}(\omega) = \frac{V}{\alpha + j\omega} \quad (32)$$

which is a continuous function for all values of ω , though $v(t)$ is discontinuous at $t = 0$.

On the left-hand side of (29) we have only the unknown function and its derivative, respectively its integral. If we assume the Fourier transform of $i(t)$ to exist and designate it as $\bar{I}(\omega)$, then we can deduce by integration by parts as in Vol. I, p. 282,

$$\mathfrak{F} \frac{di}{dt} = \int_0^{\infty} \frac{di}{dt} e^{-j\omega t} dt = -i(0^+) + j\omega \bar{I}(\omega) \quad (33)$$

For the integral term we obtain, also by integration by parts,

$$\mathfrak{F} \int i \, dt = \frac{1}{j\omega} \left(\int i \, dt \right)_{t=0^+} + \frac{1}{j\omega} \bar{I}(\omega) = \frac{1}{j\omega} [q_0 + \bar{I}(\omega)] \quad (34)$$

where q_0 is the initial charge on the condenser.

The complete Fourier transform for (29) becomes now

$$\left(j\omega L + R + \frac{1}{j\omega C} \right) \bar{I}(\omega) - Li(0^+) + \frac{q_0}{j\omega C} = \bar{V}(\omega) \quad (35)$$

Thus we have transformed the integro-differential equation (29) into an algebraic equation between transforms, which involves the initial values of the unknown function, of its first-order derivative, and of its integral. This constitutes, however, a more complete mathematical statement because it contains the arbitrary constants which normally need to be evaluated as the integration constants in the classical solution of the integro-differential equation.

We can readily solve (35) for the unknown Fourier transform $\bar{I}(\omega)$; however, to simplify the expression, let us assume zero initial values so that we obtain, with (32),

$$\bar{I}(\omega) = \frac{V}{\alpha + j\omega} \left(j\omega L + R + \frac{1}{j\omega C} \right)^{-1} = \frac{\bar{V}(\omega)}{Z(j\omega)} \quad (36)$$

which exhibits the analogy to the steady-state a-c solution of networks with $Z(j\omega)$ the a-c input impedance in complex form. Expression (36) has clearly the character of a polynomial fraction, so we can apply partial fraction expansion and identify the appropriate inverse Fourier transforms from Table 1.1. To carry through the partial fraction expansion we observe, as in connection with (15), that the transform variable ω occurs only in the combination $j\omega = p$, so that we can as well write for (36)

$$I(p) = \frac{V}{\alpha + p} \left(Lp + R + \frac{1}{Cp} \right)^{-1} \quad (37)$$

which is a real polynomial fraction in p . This can be written more conveniently

$$I(p) = \frac{V}{L} \frac{N(p)}{D(p)} \quad (38)$$

with

$$\begin{aligned} N(p) &= p \\ D(p) &= (p + \alpha) \left(p^2 + \frac{R}{L} p + \frac{1}{CL} \right) \end{aligned} \quad (39)$$

Problems with lumped-parameter circuit elements generally lead to this typical rational form in $p = j\omega$, for which the solution, i.e., the inverse Fourier transform, can be given in terms of the Heaviside expansion theorem adapted to the Fourier transform method. We can factor

$$D(p) = (p - p_1)(p - p_2)(p - p_3)$$

with

$$p_1 = -\alpha, \quad p_{2,3} = -\frac{R}{2L} \pm \left[\left(\frac{R}{2L} \right)^2 - \frac{1}{LC} \right]^{1/2}$$

Thus, by the partial fraction expansion, as in Vol. I., p. 176

$$\frac{N(p)}{D(p)} = \frac{B_1}{p - p_1} + \frac{B_2}{p - p_2} + \frac{B_3}{p - p_3} \quad (40)$$

where the coefficients

$$B_\alpha = \left(\frac{N(p)}{dD(p)/dp} \right)_{p=p_\alpha} \quad (41)$$

Combining this with pair 1 in Table 1.1,

$$\mathfrak{F}^{-1} \frac{1}{p - p_\alpha} = e^{p_\alpha t}$$

we have in typical form for the inverse of (38),

$$i(t) = \frac{V}{L} \sum_{(\alpha)} \left(\frac{N(p)}{dD(p)/dp} \right)_{p=p_\alpha} e^{p_\alpha t} \quad (42)$$

or in final form

$$i(t) = \frac{V}{L} \left[\frac{\alpha e^{-\alpha t}}{\alpha(\alpha - R/L) + 1/LC} + \frac{1}{2[(R/2L)^2 - 1/LC]^{1/2}} \left(\frac{p_2}{p_2 + \alpha} e^{p_2 t} - \frac{p_3}{p_3 + \alpha} e^{p_3 t} \right) \right] \quad (43)$$

as can readily be verified.

Obviously, (43) is also the result if we use the general inversion equation (3)

$$i(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \bar{I}(\omega) e^{i\omega t} d\omega$$

and expand $\bar{I}(\omega)$ from (36) into linear fractions.

One- and two-sided Fourier transforms. We should now add a further note of caution. The original definition of the complex Fourier integrals in (2) and (3) stipulated in time the integration from $t = -\infty$ to $t = +\infty$, yet the application to the example made us immediately change to the semi-infinite limits $t = 0^+$ to $t = +\infty$ because the voltage as defined in (31) was zero for $t < 0$. We can express this rather neatly if we add the subscript II for doubly infinite range and I for the singly infinite range of integration, thus

$$\mathfrak{F}_I f(t) = \int_{0^+}^{\infty} f(t) e^{-j\omega t} dt = F_I(j\omega) = \bar{F}_I(\omega) \quad (44)$$

$$\mathfrak{F}_{II} f(t) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt = F_{II}(j\omega) = \bar{F}_{II}(\omega) \quad (45)$$

The transform itself is a function of the imaginary variable $j\omega$ which appears as a parameter in the integrand as pointed out in (15) and (16). Because of the imaginary character of this parameter, the transform is in fact a complex function of the real frequency variable ω which is expressed by the bar in the last forms of (44) and (45).

It is important to note that the type of Fourier transform required will be defined by the applied voltage or current and that, of course, the response current or voltage must be of like kind in order to be physically consistent. For practical applications, Table 1.2 gives a list of general transform relations for these two types together with several forms of the real convolution integral in the time variable, see Vol. I, section 5.11.

For practical applications, idealized discontinuous source functions are particularly useful. We have mentioned early the unit step (8) for which the Fourier transform does not exist in the conventional sense. Using the convergence factor demonstrated in (9), we can accept as a Fourier transform

$$\mathfrak{F} S_{-1}(t) = \lim_{c \rightarrow 0} \int_0^{\infty} e^{-ct} e^{-j\omega t} dt = \frac{1}{j\omega} \quad (46)$$

It is irrelevant here whether we use the one-sided or the bilateral transform definition, because the value of unit step is zero for $t < 0$. The response to unit step is usually called the *indicial admittance* $A(t)$ and can be made the basis of the superposition theorem for arbitrary time dependence of the source function, known as the *Duhamel integral*; see Vol. I, section 2.9.

Of similar importance is the *unit impulse* which can be approximated by the analytic function

$$S_0(t) = \lim_{\alpha \rightarrow \infty} \alpha^2 t e^{-\alpha t} \quad t \geq 0 \quad (47)$$

TABLE 1.2
SHORT TABLE OF GENERAL FOURIER TRANSFORM RELATIONS

No.	Time Domain	Transfer Domain		
		\mathfrak{F}_I from (44)	\mathfrak{F}_{II} from (45)	
1	$f(t)$	$F_I(j\omega) = \bar{F}_I(\omega)$	$F_{II}(j\omega) = \bar{F}_{II}(\omega)$	Basic definitions
2	$f(t - \tau)$	$\bar{F}_I(\omega)e^{-j\omega\tau}$	$\bar{F}_{II}(\omega)e^{-j\omega\tau}$	Time shift
3	$\alpha f(\alpha t)$	$\bar{F}_I\left(\frac{\omega}{\alpha}\right)$	$\bar{F}_{II}\left(\frac{\omega}{\alpha}\right)$	Scale factor
4	$\frac{dt}{d} f(t)$	$j\omega\bar{F}_I(\omega) - f(0^+)$	$j\omega\bar{F}_{II}(\omega)$	Differentiation
5	$\int_a^t f(t) dt$	$\frac{1}{j\omega} \left[\bar{F}_I(\omega) + \left(\int_a^t f(t) dt \right)_{t=0^+} \right]$	$\frac{1}{j\omega} \bar{F}_{II}(\omega)$ $a = -\infty$	Integration
6	$e^{j\omega_0 t} f(t)$	$\bar{F}_I(\omega - \omega_0)$	$\bar{F}_{II}(\omega - \omega_0)$	Frequency shift
Convolution Integrals:				
11	$\int_a^b f_1(\tau)f_2(t - \tau) d\tau$	$[\bar{F}_I(\omega)]_1[\bar{F}_I(\omega)]_2$ $a = 0^+$ $b = t^-$	$[\bar{F}_{II}(\omega)]_1[\bar{F}_{II}(\omega)]_2$ $a = -\infty$ $b = +\infty$	General convolution (Borel theorem)
12	$e^{j\omega_0 t} \int_a^t e^{-j\omega_0 \tau} f(\tau) d\tau$	$\frac{\bar{F}_I(\omega)}{j(\omega - \omega_0)}$ $a = 0^+$	$\frac{\bar{F}_{II}(\omega)}{j(\omega - \omega_0)}$ $a = -\infty$	Linear factor
13	$f(t)$ $+ j\omega_0 e^{j\omega_0 t} \int_a^t e^{-j\omega_0 \tau} f(\tau) d\tau$	$\frac{\omega}{\omega - \omega_0} \bar{F}_I(\omega)$ $a = 0^+$	$\frac{\omega}{\omega - \omega_0} \bar{F}_{II}(\omega)$ $a = -\infty$	Linear fraction factor
Expansion Theorem:				
21	$\sum_{(\alpha)} \left[\frac{N(p)}{d D(p)/dp} \right]_{p=p_\alpha} e^{p_\alpha t}$	$\bar{F}_I(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{N(p)}{D(p)}$		Unit impulse response
$j\omega_\alpha = p_\alpha$ are the roots (distinct) of $D(p)$				

as discussed in Vol. I, p. 194. The general shape can be inferred from Fig. 1.10 where the curves are drawn for different values of α . Obviously, in order to have an impulse of duration $\tau < 10^{-6}$ sec, we must have $\alpha\tau = 4$, or $\alpha > 4 \times 10^6$ 1/sec. Expression (47) is normalized such that the integral has value unity,

$$\int_0^\infty \alpha^2 t e^{-\alpha t} dt = 1$$

In the limit, $S_0(t)$ represents a spike of infinite height and infinitesimal duration at $t = 0^+$, i.e., on the positive side of the time abscissa, yet with the product value unity. The Fourier transform is therefore

$$\mathfrak{F}S_0(t) = \int_0^\infty S_0(t)e^{-j\omega t} dt = 1 \quad (48)$$

because over the infinitesimal duration the exponential factor assumes the value unity. The response to unit impulse is therefore given directly by the inverse transform of the network impedance or admit-

tance function. For lumped-parameter networks it can be evaluated by the *Heaviside expansion theorem* which is shown in the last line of Table 1.2.

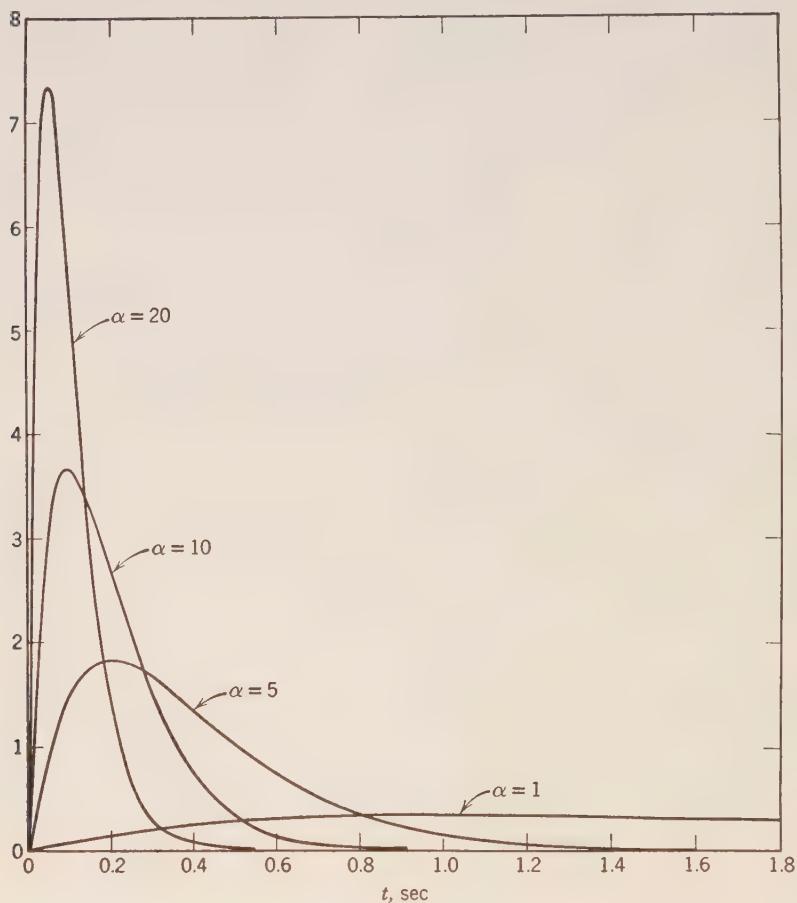


Fig. 1.10. Plot of the function $\alpha^2 t e^{-\alpha t}$ as approximation to unit impulse $S_0(t)$.

1.4 The Laplace Transform Method

The Fourier transform method has two disadvantages in practical applications: the time integral in the direct transform often does not converge without providing an extra convergence factor as in (9), and the inverse transform is a real integral in the frequency variable which often requires a cautious approach, even if it can be evaluated by conventional methods. Both disadvantages disappear at once if we turn to the Laplace transform method.

The direct Laplace transform. We shall generally use the one-sided direct Laplace transform which is particularly useful in transient analysis, assuming the source functions suddenly applied at $t = 0$ and being zero for $t < 0$. Given the time function $f(t)$, we define the Laplace transform

$$\mathcal{L}f(t) = \int_{t=0^+}^{\infty} f(t)e^{-pt} dt = F(p) \begin{cases} p = \delta + j\omega \\ \delta > 0 \end{cases} \quad (49)$$

This incorporates right into the complex exponential an appropriate convergence factor which will guarantee the existence of Laplace transforms for practically all the functions of possible interest to us. Compared with the definition of the one-sided Fourier transform (44), we see at once that

$$\mathcal{L}f(t) \equiv \mathfrak{F}_I[f(t)e^{-\delta t}] \quad (50)$$

We observe that, if

$$\lim_{\delta \rightarrow 0} F(p) \rightarrow F(j\omega) = \bar{F}(\omega)$$

exists, it must indeed be the Fourier transform, at least in the Abel-Poisson sense as outlined in connection with (9) and (10).

Actually, (49) defines the Laplace transform only for the value δ or $p = \delta + j\omega$ for which the integration has been performed. Conceiving of p , however, as a complex variable defining the complex p -plane, we can extend the meaning of $F(p)$ to be a function of the general complex variable p and to exist in the entire p -plane. We can thus examine $F(p)$ like any function of a complex variable, locate its singularities, and, in general, describe its characteristics for all values of p . $F(p)$ becomes then an associated complex function to $f(t)$, an “*image*” of the real time function in the complex p -domain.

For unit step, defined in (8) with these equivalent symbols

$$1 = H(t) = S_{-1}(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases} \quad (51)$$

we obtain quite readily because of the assumption $\text{Re } p > 0$

$$\mathcal{L}S_{-1}(t) = \int_{0^+}^{\infty} e^{-pt} dp = \frac{1}{p} \quad (52)$$

It is worth noting that the Fourier transform (46) could be obtained only by using a special convergence factor, which is here inherent. We also see readily that for $\delta \rightarrow 0$ we obtain from (52) the Fourier transform (46), thus justifying the procedure used there. From this and similar examples one can argue that the modern Laplace transform

TABLE 1.3
SHORT TABLE OF RATIONAL LAPLACE TRANSFORMS

No.	Time Function $f(t)$, t -Domain	Laplace Transform $F(p)$, p -Domain
1	$1 \equiv S_{-1}(t)$	$\frac{1}{p}$
2	$\frac{t}{\tau} \quad t \geq 0$	$\frac{1}{p^2 \tau}$
3	$\frac{t^n}{n!} \quad t \geq 0$	$\frac{1}{p^{n+1}}$
4	$e^{-\delta t} \quad t > 0$	$\frac{1}{p + \delta}$
5	$(1 - e^{-\delta t}) \quad t \geq 0$	$\frac{\delta}{p(p + \delta)}$
6	$e^{j\omega t} \quad t > 0$	$\frac{1}{p - j\omega}$
7	$\sin \omega t \quad t \geq 0$	$\frac{\omega}{p^2 + \omega^2}$
8	$\cos \omega t \quad t > 0$	$\frac{p}{p^2 + \omega^2}$
11	$\delta t e^{-\delta t} \quad t \geq 0$	$\frac{\delta}{(p + \delta)^2}$
12	$\frac{(\delta t)^n}{n!} e^{-\delta t} \quad t \geq 0$	$\frac{\delta^n}{(p + \delta)^{n+1}}$
13	$\frac{1}{\delta_2 - \delta_1} (e^{-\delta_1 t} - e^{-\delta_2 t}) \quad t \geq 0$	$\frac{1}{(p + \delta_1)(p + \delta_2)}$
14	$e^{-\delta t} \sin \Omega t \quad t \geq 0$	$\frac{\Omega}{(p + \delta)^2 + \Omega^2}$
15	$e^{-\delta t} \cos \Omega t \quad t > 0$	$\frac{p + \delta}{(p + \delta)^2 + \Omega^2}$
16	$\frac{1}{2\omega} t \sin \omega t \quad t \geq 0$	$\frac{p}{(p^2 + \omega^2)^2}$
17	$t \cos \omega t \quad t \geq 0$	$\frac{p^2 - \omega^2}{(p^2 + \omega^2)^2}$

Singular Pairs:

21	$S_{-1}(t) = \lim_{\alpha \rightarrow \infty} [1 - (1 + \alpha t)e^{-\alpha t}], t \geq 0$	$\frac{1}{p} = \lim_{\alpha \rightarrow \infty} \frac{\alpha^2}{p(p + \alpha)^2}$
22	$S_0(t) = \lim_{\alpha \rightarrow \infty} [\alpha^2 t e^{-\alpha t}], t \geq 0$	$1 = \lim_{\alpha \rightarrow \infty} \frac{\alpha^2}{(p + \alpha)^2}$
23	$S_1(t) = \lim_{\alpha \rightarrow \infty} [\alpha^2 (1 - \alpha t)e^{-\alpha t}], t \geq 0$	$p = \lim_{\alpha \rightarrow \infty} \frac{p\alpha^2}{(p + \alpha)^2}$

is the broader and more basic concept and the classical Fourier transform represents a special version of it.

Table 1.3 gives a short list of rational Laplace transforms most frequently occurring in connection with conventional source functions and simple lumped, passive circuit problems. It also contains the important singular source functions and their analytic approximations: unit step $S_{-1}(t)$, unit impulse $S_0(t)$, and unit doublet $S_1(t)$ with the notations introduced by Campbell-Foster.¹ A detailed discussion of these singular functions is given in Vol. I, section 5.6. The unit impulse is also treated above in section 1.3 and progressive approximation by an analytic function is shown in Fig. 1.10.

The inverse Laplace transform. To formulate the inverse Laplace transform, we start with the basic Fourier integral (3)

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t} d\omega$$

and observe that ω is the integration variable. Replacing $j\omega$ by $p = \delta + j\omega$ as required by (49), we must take

$$dp = j d\omega$$

i.e., integrate parallel to the imaginary axis of the p -plane, and, actually, we must integrate with $\text{Re } p = \delta > 0$ as presumed for (49). We thus obtain

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(p) e^{pt} dp = \mathcal{L}^{-1}F(p) \quad (53)$$

where c must be so chosen that $f(t) = 0$ for $t < 0$ since this had been the basis for (49). Again, if we can let $\delta \rightarrow 0$ and the integrand exists there for all values ω , then the inverse Laplace transform reduces to the inverse Fourier transform. The great advantage of (53) is, however, that the integration can be interpreted as integration in the complex p -plane, with the integrand $F(p)e^{pt}$ usually a regular function of the complex variable p except for denumerable singularities. We can bring to the evaluation of (53) all the background of function theory which is of invaluable assistance!

It is perhaps not amiss to emphasize that, unless the Laplace transform $F(p)$ continuously approaches $F(j\omega)$ as $p \rightarrow j\omega$ or $\delta \rightarrow 0$ for all values of ω ,

$$\mathcal{L}^{-1}F(p) \neq \mathcal{F}^{-1}F(j\omega)$$

in general. Conversely, having given $F(j\omega)$, it is not assured that we can simply substitute $j\omega$ by p and consider $F(p)$ as the analytic function

taking the values $F(j\omega)$ on the imaginary axis.* Actually, by Cauchy's integral theorem (18), Appendix 5, we can construct

$$F(p) = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \frac{F(j\omega)}{j\omega - p} d(j\omega)$$

if $F(j\omega)$ is regular along the entire imaginary axis as well as everywhere to the right of it. Should $F(j\omega)$ have any singularity on the imaginary axis, then this integral equation cannot be used. One might shift a removable singularity from the imaginary axis slightly to the left, i.e., use essentially the convergence factor discussed in (9) and (10) and accept convergence in the Abel-Poisson sense rather than insist on conventional convergence.

If the function $F(p)$ is rational, it can be expressed as a fraction of two polynomials, and expanded into linear fractions

$$F(p) = \frac{N(p)}{D(p)} = \sum_{(\alpha)} \frac{B_{\alpha}}{p - p_{\alpha}} \quad (54)$$

as demonstrated in (40) and (41). The p_{α} are the roots of $D(p)$, assumed to be distinct. Thus, the function $F(p)$ possesses poles of first order at the points p_{α} in the complex plane, see Appendix 5, section 2, and is regular at $p \rightarrow \infty$ if the degree of $N(p)$ is less than that of $D(p)$. The evaluation of the integral (53) is now performed easiest by utilizing the Cauchy fundamental integral theorem as outlined in detail in Vol. I, pp. 205–208. Referring to Fig. 1.11, which is the same as Fig. 5.11 of Vol. I, we first demonstrated that the contributions of the straight line path C' below A and above B are negligibly small. We then replaced the path AB for $t > 0$ by the closed path $ABDEFA$ minus the closing arc $BDEFA$. The latter contributes vanishingly little, the former contributes the sum of the residues at the simple poles which is given by (25), Appendix 5, as

$$2\pi j \sum_{(\alpha)} B_{\alpha} e^{p_{\alpha} t}$$

For $t < 0$ we demonstrated that closure of the path to the right gives a zero result for the desired integral as required by the fact that $f(t) = 0$ for $t < 0$. Indeed, the last requirement demands that the distance c be so chosen that no singularities of $F(p)$ lie to the right of it. We thus have the simple result

* See also E. A. Guillemin, *The Mathematics of Circuit Analysis*, p. 548, Wiley, New York, 1949.

$$f(t) = \sum_{(\alpha)} \left(\frac{N(p)}{dD(p)/dp} \right)_{p=p_\alpha} e^{p_\alpha t} \tag{55}$$

identical with line 11 of Table 1.4 and recognized as a general form of Heaviside's expansion theorem.

The standard form of the Heaviside expansion theorem results if a step voltage $V1$ with transform V/p from (52) is applied to a network

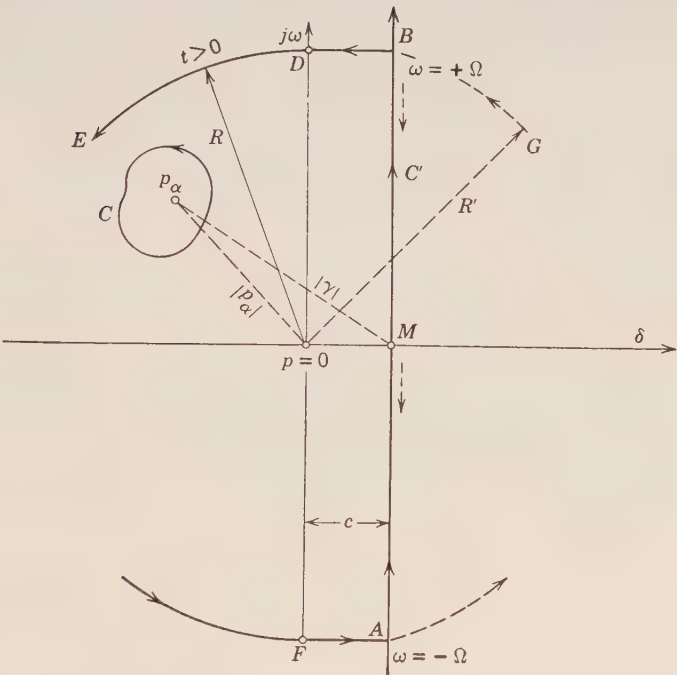


Fig. 1.11. Cauchy's integral relation and the inverse Laplace transform.

composed only of lumped parameter, passive, bilateral elements. In such cases the current transform would appear as

$$I(p) = \frac{1}{Z(p)} \frac{V}{p} \tag{56}$$

where $Z(p)$ is the parametric impedance defined as $D(p)/N(p)$. The extra root $p = 0$ of the denominator in (56) gives the steady-state response which is usually separated so that we get as in Vol. I, (5.139),

$$i(t) = V \left\{ \frac{N(0)}{D(0)} + \sum_{(\alpha)} \left[\frac{N(p)}{p[dD(p)/dp]} \right]_{p=p_\alpha} e^{p_\alpha t} \right\} = V A(t) \tag{57}$$

where $A(t)$ is the *indicial admittance*. Analogously, if we find a voltage transform for a step current $I1$ impressed upon a network, we can define an *indicial impedance*. In general we shall refer to either as *indicial response* $R(t)$ of the network because it is a basic characteristic for the transient behavior. $R(t)$ is listed in Table 1.4, line 12.

TABLE 1.4
SHORT TABLE OF GENERAL LAPLACE TRANSFORM RELATIONS

No.	Time Domain	Transfer Domain	Designation
1	$f(t), t > 0$	$F(p)$	Basic correspondence
1a	$f(0^+)$	$\lim_{p \rightarrow \infty} pF(p)$	Initial value
1b	$f(\infty)$	$\lim_{p \rightarrow 0} pF(p)$	Final value
2	$f(t - \tau), t > \tau$	$F(p)e^{-p\tau}$	Delay
3	$\frac{d}{dt}f(t)$	$-f(0^+) + pF(p)$	Derivative
4	$\int f(t) dt = g(t)$	$\frac{1}{p}g(0^+) + \frac{1}{p}F(p)$	Integral
4a	$\int_{0^+}^t f(t) dt$	$\frac{1}{p}F(p)$	Definite integral
5	$f(t)e^{\gamma t}$	$F(p - \gamma)$	Shifting theorem
5a	$f(t)e^{j\omega t}$	$F(p - j\omega)$	A-c shifting
6	$f(at)$	$\frac{1}{a}F\left(\frac{p}{a}\right)$	Scale factor
6a	$\omega f(\omega t)$	$F\left(\frac{p}{\omega}\right)$	Frequency scale
Response Functions:			
11	$D(t) = \sum_{(\alpha)} B_{\alpha} e^{p_{\alpha} t}$	$\frac{N(p)}{D(p)} = \sum \frac{B_{\alpha}}{p - p_{\alpha}}$	Expansion theorem (unit impulse response) (55)
	$B_{\alpha} = \left(\frac{N(p)}{dD(p)/dp} \right)_{p=p_{\alpha}}$	p_{α} roots (distinct) of $D(p)$	
12	$R(t) = \frac{N(0)}{D(0)} + \sum_{(\alpha)} \frac{B_{\alpha}}{p_{\alpha}} e^{p_{\alpha} t}$ (Indicial response)	$\frac{N(p)}{pD(p)} = \frac{1}{p} \sum \frac{B_{\alpha}}{p - p_{\alpha}}$	Heaviside expansion theorem (unit step response) (57)
13	$\bar{C}(t) = \frac{N(j\omega)}{D(j\omega)} e^{j\omega t} + \sum_{(\alpha)} \frac{B_{\alpha}}{p_{\alpha} - j\omega} e^{p_{\alpha} t}$	$\frac{N(p)}{(p - j\omega)D(p)} = \frac{1}{p - j\omega} \sum \frac{B_{\alpha}}{p - p_{\alpha}}$	Complex a-c expansion (unit cisoid response) (61)

The general a-c source function (voltage or current) with unit amplitude is written in complex form

$$e^{j\omega t} = \cos \omega t + j \sin \omega t = \text{cis } \omega t$$

where the last form is taken from Campbell (see Campbell-Foster,^{E1} p. 5) and is called "cisoid," indicating symbolically the complex combination of cos and sin functions. The Laplace transform of it is found readily

$$\mathcal{L}e^{j\omega t} = \int_{0^+}^{\infty} e^{j\omega t} e^{-pt} dt = \frac{1}{p - j\omega} \quad (58)$$

which is also listed in Table 1.3. If we use this instead of the unit step transform in (56) we can derive a complex expansion theorem for unit cisoid response $\tilde{C}(t)$ which is listed in Table 1.4, line 13. To obtain the physical solution we need to multiply $\tilde{C}(t)$ by the phasor of the source function and take either the real or the imaginary part of the product, depending upon the original definition of the source function. Let us assume we apply the voltage

$$V_m \sin(\omega t + \psi) = I_m(Ve^{j\omega t}) \quad (59)$$

to a lumped-parameter, passive network. The phasor is thus

$$V = V_m e^{j\psi}$$

and the complete current response in complex notation

$$\tilde{i}(t) = V\tilde{C}(t) \quad (60)$$

so that the physical solution becomes

$$i(t) = \text{Im } \tilde{i}(t) = \text{Im } [V\tilde{C}(t)] \quad (61)$$

The appropriate modification for a current source is simple since the response function $\tilde{C}(t)$ has not been specified as to physical significance.

Example. Let us illustrate the use of the Laplace transform method by a simple circuit problem. Suppose we apply a current pulse of the shape

$$i_0(t) = I\alpha t e^{-\alpha t} \quad t \geq 0 \quad (62)$$

corresponding to the right-hand half of Fig. 1.1 and shown in Fig. 1.12a, to the parallel G - L circuit in Fig. 1.12c. The node-pair equation for the output voltage $v_o(t)$ can be read from the figure as

$$i_0(t) = \frac{1}{L} \int v_o(t) dt + Gv_o(t) \quad (63)$$

To take the Laplace transform of this equation, we multiply each term by e^{-pt} and integrate with respect to time in accordance with (49). The transform of the current is from Table 1.3, line 16

$$\mathcal{L}i_0(t) = I \frac{\alpha}{(p + \alpha)^2}$$

The transform of the voltage is unknown; let us call it $V(p)$. For the time integral we might apply integration by parts, choosing $e^{-pt} dt$ as

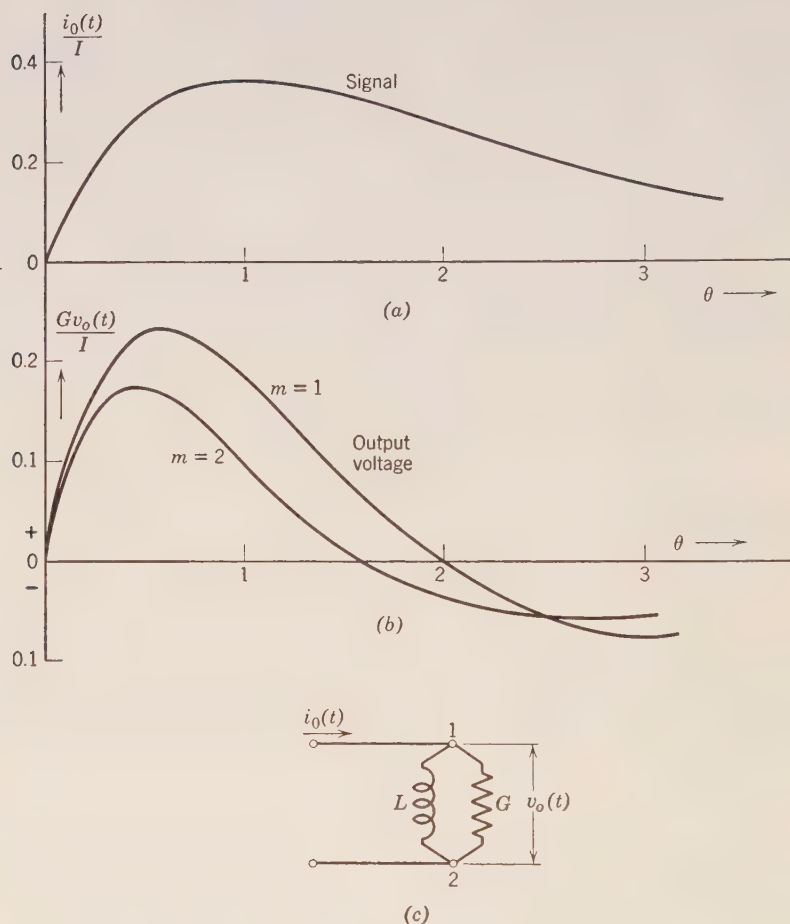


Fig. 1.12. Response (b) of parallel G - L circuit (c) to signal current (a).

the differential factor,

$$\int_{0^+}^{\infty} \left(\int v_o(t) dt \right) e^{-pt} dt = -\frac{1}{p} e^{-pt} \int v_o(t) dt \Big|_0^{\infty} + \int_{0^+}^{\infty} \frac{1}{p} v_o(t) e^{-pt} dt$$

The first term vanishes at the upper limit if $\int v_o(t) dt$ exists as $t \rightarrow \infty$. The second term can be identified as $p^{-1}V(p)$, so that the total result is

$$\mathcal{L} \int v_o(t) dt = \frac{1}{p} v_o(0^+) + \frac{1}{p} V(p) \quad (64)$$

which also appears as a general relation in Table 1.4, line 4. Collecting the terms, we get for the transform of (63)

$$\left(G + \frac{1}{pL}\right) V(p) = -\frac{1}{pL} \left(\int v_o(t) dt \right)_{0^+} + \frac{\alpha}{(p + \alpha)^2} I \quad (65)$$

which is an algebraic relationship between the known and unknown Laplace transforms.

Because p is a complex variable, (65) is also a simple linear equation between functions of the complex variable p which can be solved at once

$$V(p) = -\frac{(\int v_o(t) dt)_{0^+}}{pL[G + 1/(pL)]} + \frac{\alpha I}{(p + \alpha)^2[G + 1/(pL)]} \quad (66)$$

We see that the voltage transform is the sum of two parts, one depending only upon the initial value of the time integral of the voltage drop v_o across the parallel circuit elements, the other depending only upon the applied source current. We could obviously have solved independently for each part, i.e., assume that the initial energy stored in the circuit was zero at $t = 0$; then only the response to the source current would develop. On the other hand, if $\int v_o dt$ is not zero initially, we find the additional solution as the decay of the initial energy in the circuit elements entirely uninfluenced by the response to the source current. These observations illustrate the general validity of superposition in linear transient problems. Without loss of generality, we can therefore restrict ourselves to initially de-energized conditions if we prefer that for reasons of simplicity.

The inverse Laplace transform of the first part in (66) is found directly from Table 1.3, line 11

$$-\frac{1}{GL} \left(\int v_o(t) dt \right)_{0^+} e^{-\frac{t}{GL}}$$

and represents the exponential decay of the initial current in the inductance L by dissipation in the conductance G .

The inverse transform of the second part is actually more significant. Let us therefore disregard the first part and rewrite (66)

$$V(p) = \frac{\alpha I}{G} \frac{p}{(p + \alpha)^2(p + \gamma)} \quad (67)$$

where $\gamma = 1/LG$. The inverse transform is then by (53)

$$v_o(t) = \frac{\alpha I}{G} \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{p}{(p + \alpha)^2(p + \gamma)} e^{pt} dp \quad (68)$$

The rational part of this integrand has a pole of second order at $p = -\alpha$. Even so, we can still apply the sum of residues to evaluate

this integral. On the basis of Cauchy's fundamental integral theorem we have from Appendix 5, section 4, for the residue at the second-order pole

$$\text{Residue } \frac{h(p)}{(p + \alpha)^2} = 2\pi j \left(\frac{d}{dp} h(p) \right)_{p=-\alpha} \quad (69)$$

where in our case

$$h(p) = \frac{p}{p + \gamma} e^{pt}$$

so that

$$\frac{d}{dp} h(p) = \left(\frac{\gamma}{(p + \gamma)^2} + \frac{pt}{p + \gamma} \right) e^{pt} \quad (70)$$

The total sum of residues is now (69) and the residue at the first-order pole $p = -\gamma$, giving for (68)

$$v_o(t) = \frac{\alpha I}{G} \left(\frac{\gamma}{(\gamma - \alpha)^2} (e^{-\alpha t} - e^{-\gamma t}) - \frac{\alpha t}{\gamma - \alpha} e^{-\alpha t} \right) \quad (71)$$

The character of the response will depend on the relative magnitudes of α and γ . We might introduce

$$\alpha t = \theta, \quad \gamma/\alpha = m, \quad \gamma t = m\theta$$

and thus obtain

$$v_o(\theta) = \frac{I}{G} \left(\frac{m}{(m - 1)^2} (e^{-\theta} - e^{-m\theta}) - \frac{\theta}{m - 1} e^{-\theta} \right)$$

which has been plotted in Fig. 1.12*b* as a function of θ for the parameter values $m = 1$ and $m = 2$. As m becomes very large, the output voltage approximates the derivative of the input current $i_0(t)$ as one can readily check from the physical conditions.

1.5 Convolution and Superposition Integrals

Frequently the task of finding the inverse Laplace transform can be made simpler by the use of superposition integrals which take advantage of the linear relationships to which we have restricted our attention.

Step summation. We assume that we know the indicial response of a network, e.g., the indicial admittance $A(t)$. If we then have given an arbitrary voltage function $v(t)$ as in Fig. 1.13, we may approximate it by a *staircase* function, i.e., a series of finite steps. We can assume $v(0^+)$, the first step, as well as all subsequent steps δv_α to be step func-

tion voltages, so that $v(t)$ can be expressed as the sum

$$v_n = v(0^+)S_{-1}(t) + \sum_{\alpha=1}^n \delta v_{\alpha} S_{-1}(t - \alpha \delta t) \quad (72)$$

where n is the largest integer so chosen that

$$t - n \delta t > 0$$

if t is the continuous time variable. The δt are here all taken as alike, though in a more general representation one could also choose the time

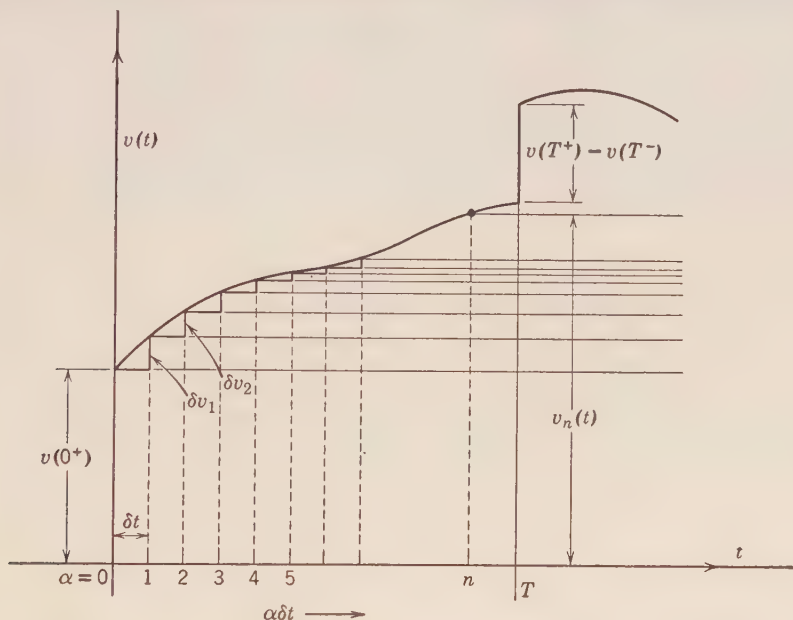


Fig. 1.13. Representation of voltage $v(t)$ as staircase function by horizontal slicing.

intervals larger where the variation of $v(t)$ is less rapid, and smaller where $v(t)$ varies more rapidly. The increments of the voltage can be directly read from the graph of the voltage. The current response for the staircase approximation is simply

$$i(t) = v(0^+)A(t)S_{-1}(t) + \sum_{\alpha=1}^n \delta v_{\alpha} A(t - \alpha \delta t)S_{-1}(t - \alpha \delta t) \quad (73)$$

where $i(t)$ is made up of curve segments, one for each δt , as defined by $A(t)$ and the $A(t - \alpha \delta t)$, respectively. We have the superposition of the discrete step voltage responses up through step n .

TABLE 1.5
CONVOLUTION AND SUPERPOSITION INTEGRALS

Components			Product		Inverse Transform		Designation
No.	$F_1(p)$	$\mathcal{L}^{-1}F_1(p)$	$F_2(p)$	$\mathcal{L}^{-1}F_2(p)$	$F(p)$	$\mathcal{L}^{-1}F(p)$	
1	$F_1(p)$	$f_1(t)$	$F_2(p)$	$f_2(t)$	F_1F_2	$\int_{\tau=0^+}^{t^-} f_1(\tau)f_2(t-\tau) d\tau$	General real convolution (Borel theorem) (90)
2	$Q(p)$	$q(t)$	$W(p)$	$D(t)$	$Q(p)W(p)$	$\int_{\tau=0^+}^{t^-} q(\tau)D(t-\tau) d\tau$	Giorgi superposition theorem (82)
3	$pQ(p) = q(0^+) + \mathcal{L} \frac{dq}{dt}$		$\frac{1}{p} W(p)$	$R(t)$	$pQ(p) \frac{1}{p} W(p)$	$q(0^+)R(t) + \int_{\tau=0^+}^{t^-} \frac{dq(\tau)}{d\tau} R(t-\tau) d\tau$	Duhamel integral (74)
11	$F(p)$	$f(t)$	$\frac{1}{p-\gamma}$	$e^{\gamma t}$	$\frac{F(p)}{p-\gamma}$	$e^{\gamma t} \int_{\tau=0^+}^{t^-} f(\tau)e^{-\gamma\tau} d\tau$	Exponential convolution
12	$W(p)$	$D(t)$	$\frac{1}{p-j\omega}$	$e^{j\omega t}$	$\frac{W(p)}{p-j\omega}$	$e^{j\omega t} \int_{\tau=0^+}^{t^-} D(\tau)e^{-j\omega\tau} d\tau$	Complex a-c solution from impulse response
13	$\frac{1}{p} W(p)$	$R(t)$	$\frac{p}{p-j\omega}$	$S_0(t) + j\omega e^{j\omega t}$	$\frac{W(p)}{p-j\omega}$	$R(t) + j\omega e^{j\omega t} \int_{\tau=0^+}^{t^-} R(\tau)e^{-j\omega\tau} d\tau$	Complex a-c solution from unit step response
14	$W(p)$	$D(t)$	$\frac{1}{p^2}$	$tS_{-1}(t)$	$\frac{W(p)}{p^2}$	$tR(t) - \int_{\tau=0^+}^{t^-} D(\tau)\tau d\tau = \int_{\tau=0^+}^{t^-} R(\tau) d\tau$	Linear time rise response

Form (72) represents the voltage function in terms of horizontal slices. If we apply the same representation to $A(t)$ for ease of numerical computation, then $i(t)$ becomes also a staircase function. On the other hand, we can let the intervals become differentially small $\delta t \rightarrow d\theta$, then $\alpha \delta t \rightarrow \theta$, and

$$\delta v_\alpha \rightarrow \left(\frac{dv}{dt} \right)_{t=\theta} d\theta$$

so that (73) goes over into the Duhamel integral, see Vol. I, p. 70, as well as Table 1.5

$$i(t) = v(0^+)A(t) + \int_{\theta=0}^{\theta=t} A(t-\theta) \frac{dv(\theta)}{d\theta} d\theta \quad (74)$$

where the unit step symbols have been left off as unnecessary. At any finite discontinuity of the voltage such as at $t = T$ in Fig. 1.13, we need to insert the finite response similar to the first term in (74). We would thus obtain for values of $t > T$,

$$\begin{aligned} i(t) = v(0^+)A(t) + \int_{\theta=0}^{\theta=T} A(t-\theta) \frac{dv(\theta)}{d\theta} d\theta + \delta v_T A(t-T) \\ + \int_{\theta=T}^{\theta=t} A(t-\theta) \frac{dv(\theta)}{d\theta} d\theta \quad (75) \end{aligned}$$

Pulse sampling. A rather different approximation principle is by means of continuous *pulse sampling*. We can represent the value

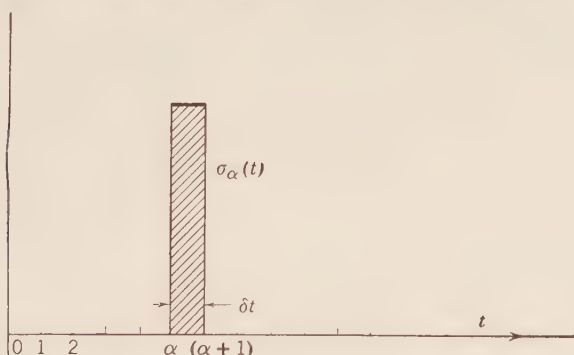


Fig. 1.14. Sampling pulse.

of the voltage for the interval $\alpha \delta t < t < (\alpha + 1) \delta t$ by multiplying the voltage value v_α at the beginning of the interval δt with the pulse shown in Fig. 1.14, namely

$$\sigma_\alpha(t) = S_{-1}(t - \alpha \delta t) - S_{-1}[t - (\alpha + 1) \delta t] \quad (76)$$

The amplitude is kept at unity so that the total voltage $v(t)$ for any instant t becomes the single term

$$v(t) = v_n = v(n \delta t) \sigma_n(t) \quad (77)$$

where n again is the largest integer for which still

$$t - n \delta t > 0$$

This gives again a staircase approximation of the voltage function as

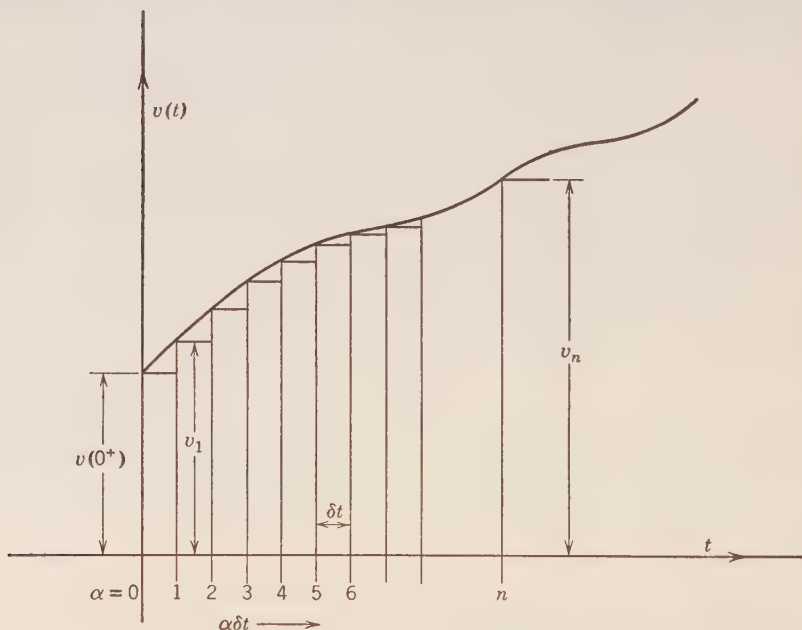


Fig. 1.15. Representation of voltage $v(t)$ as staircase function by vertical slicing, i.e., by sampling pulses.

shown in Fig. 1.15 but here in terms of contiguous rectangular pulses, each of the shape (76), i.e., of columnar type.

The Laplace transform of the individual pulse is

$$\mathcal{L}\sigma_\alpha(t) = \frac{1}{p} (e^{-p\alpha\delta t} - e^{-p(\alpha+1)\delta t}) = \frac{e^{-p\alpha\delta t}}{p} (1 - e^{-p\delta t}) \quad (78)$$

and that of the staircase voltage (77), integrated from $t = 0$ to $t = \infty$,

$$\mathcal{L}v(t) = \frac{1 - e^{-p\delta t}}{p} \sum_{\alpha=0}^{\infty} v_\alpha e^{-p\alpha\delta t} \quad (79)$$

where we used v_α for the value $v(t)$ at $t = \alpha \delta t$. The first factor in (79) is the Laplace transform of the sampling pulse itself; it appears as the difference of the two unit step components. The transform solution for the current can well be taken as (79) divided by the parametric impedance $Z(p)$,

$$\mathcal{L}i(t) = \frac{1 - e^{-p\delta t}}{pZ(p)} \sum_{\alpha=0}^{\infty} v_\alpha e^{-p\alpha\delta t} \quad (80)$$

The inverse transform of the factor in front of the summation is the difference

$$A(t) - A(t - \delta t)$$

where $A(t)$ is again the indicial admittance as a continuous time function. The summation now introduces time shifts of $\alpha \delta t$, so that the inverse transform becomes

$$i_n = \sum_{\alpha=0}^{\alpha=n-1} v_\alpha [A(t - \alpha \delta t) - A(t - \alpha \delta t - \delta t)] + v_n A(t - n \delta t)$$

where $n \delta t < t < (n + 1) \delta t$. Each indicial admittance is zero for $t < \alpha \delta t$ or $t < (n + 1) \delta t$ but extending over all time afterwards.

This form lends itself singularly well to numerical computations if we represent $A(t)$ also as a columnar staircase function, i.e., define in analogy to (77)

$$A(t) = A_n = A(n \delta t) \sigma_n(t)$$

Introducing this with the appropriate adjustments into the current expression, we obtain

$$i_n = \sum_{\alpha=0}^{\alpha=n-1} v_\alpha [A(\overline{n - \alpha} \delta t) \sigma_{n-\alpha}(t) - A(\overline{n - \alpha - 1} \delta t) \sigma_{n-\alpha-1}(t)] + v_n A(t - n \delta t) \sigma_n(t) \quad (81)$$

which is a constant value valid for $n \delta t < t < (n + 1) \delta t$. Obviously, v_α is zero for $\alpha < 0$, and $A(\overline{n - \alpha} \delta t)$ is zero for $n - \alpha < 0$, because $A(t)$ is an indicial admittance. Thus we need to form the products of voltage values progressing as shown in Fig. 1.16 from left to right with indicial admittance values progressing from right to left. This is the *convolution summation* for numerical or graphical evaluation of responses to difficult source functions; it is based upon the continuous pulse sampling and leads to a staircase representation of the response.

We could let the pulse duration decrease indefinitely, by letting $\delta t \rightarrow d\theta$. In (80), the first factor then becomes

$$\lim_{\delta t \rightarrow d\theta} \frac{1 - e^{-p\delta t}}{p} \rightarrow d\theta$$

The inverse transform of $Z(p)^{-1}$ is the unit impulse response $D(t)$ in line 11, Table 1.4, and the summation goes over into the integral with

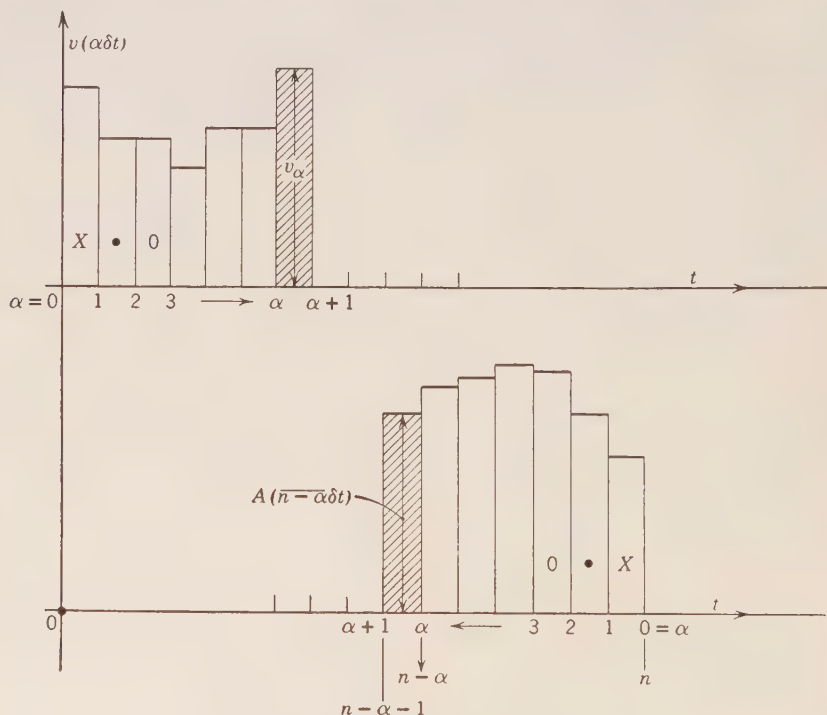


Fig. 1.16. Convolution summation on sampling pulse basis.

$\alpha \delta t \rightarrow \theta$, $v_\alpha \rightarrow v(\theta)$, so that with the appropriate delay indicated by the exponential we obtain

$$i(t) = \int_{\theta=0}^{\theta=t} v(\theta) D(t - \theta) d\theta \quad (82)$$

This is another special form of the general superposition principle, introduced by Giorgi, see Vol. I, p. 228. This theorem is also listed in Table 1.5.

On the other hand, we can apply similar relations to the discrete pulse sampling as often used in communication systems.* Assume

* See H. S. Black, *Modulation Theory*, chapter 4, Van Nostrand, New York, 1953.

that the individual pulses have the duration δt as in Fig. 1.14, but that they occur at equal intervals $\beta \delta t$. The pulse numbered α has then the form

$$\sigma_{\alpha}(t) = S_{-1}(t - \alpha\beta \delta t) - S_{-1}[t - (\alpha\beta + 1) \delta t] \quad (83)$$

With the appropriate modification, the current response will now be given by

$$i_n(t) = \sum_{\alpha=0}^{n-1} v_{\alpha} \{ A(t - \alpha\beta \delta t) - A[t - (\alpha\beta + 1) \delta t] \} + v_n A(t - n \delta t),$$

$$n\beta \delta t < t < (n + 1)\beta \delta t \quad (84)$$

where t is a continuous variable and must lie in the interval specified. Obviously, if the time interval between pulses $(\beta - 1) \delta t$ is large enough so that the response to the pulse at the beginning of the period has practically vanished by the end of the period, then the summation reduces to a single term,

$$i_n(t) \approx v_n \{ A(t - n\beta \delta t) - A[t - (n\beta + 1) \delta t] \}$$

which is particularly useful for communication purposes if economically permissible. Clearly, the individual pulse response is a direct measure of the initiating voltage sample.

Impulse sampling. Finally, we can let the sampling pulse in Fig. 1.14 become of infinitesimal duration. In order to preserve the proper relationships of the physical quantities, we might start from (77) and modify it to read for any instant of time t

$$v(t) = v(t) \delta t \frac{\sigma(t)}{\delta t} \quad (85)$$

The limit

$$\lim_{\delta t \rightarrow 0} \frac{\sigma(t)}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{S_{-1}(t) - S_{-1}(t - \delta t)}{\delta t} \rightarrow S_0(t)$$

expresses the fact that we let the duration of the pulse become infinitesimal but at the same time let its amplitude grow beyond all limits so that the total area under the resulting impulse is unity

$$\int_{t=0}^{t>0} S_0(t) dt = 1 \quad (86)$$

as already defined in a different approach in (47) and more thoroughly discussed in Vol. I, section 4.6, p. 138. Because the time integral gives the unit value, $S_0(t)$ itself must be expressed in inverse time units, e.g., in sec^{-1} . Therefore, we find in (85) that we actually need to use

the voltage value multiplied by δt to preserve the physical dimensions. We can write (85) in the two forms,

$$v(t) = M_v S_0(t) = v(t) S_0(t) dt \quad (87)$$

Because of (86), the second form is simply the voltage value $v(t)$ multiplied by unity but represented as an infinite spike. The first form associates the time element δt with the voltage; in practical computation we can assume M_v to represent the value of the integral

$$M_v = \int_{\theta=t}^{\theta>t} v(\theta) d\theta \quad (88)$$

over the actual voltage spike. M_v then gives the finite value of the area under the impulse and $S_0(t)$ defines its shape in ideal form. Obviously, M_v is measured in volt-sec.

The current response to a single impulse occurring at time θ is then given by

$$i(t) = M_v(\theta) D(t - \theta) \quad t \geq \theta \quad (89)$$

The continuous impulse sampling of the voltage leads at once to the integral (82), each individual contribution being infinitesimally small. For practical applications we have to keep in mind that $S_0(t)$ is an abstract concept and that the approximations discussed in Vol. I, sections 4.6 and 5.6, give a more readily comprehensible solution. After all, the physical sources will produce voltage forms more closely approached by the approximating time functions than by the mathematical constructs. Interesting discussions of the impulse functions are found in Van der Pol-Bremmer,^{D15} chapter V.

General convolution integrals. As demonstrated in Vol. I, section 5.11, if we have a Laplace transform that can be resolved into a product of two simpler Laplace transforms

$$F(p) = F_1(p)F_2(p)$$

where the inverse Laplace transforms for the two factors are known to be, respectively,

$$\mathcal{L}^{-1}F_1(p) = f_1(t), \quad \mathcal{L}^{-1}F_2(p) = f_2(t)$$

we find for the inverse transform of the product

$$\begin{aligned} \mathcal{L}^{-1}F(p) &= \int_{\tau=0+}^{t-} f_1(\tau) f_2(t - \tau) d\tau \\ &= \int_{\tau=0+}^{t-} f_1(t - \tau) f_2(\tau) d\tau \end{aligned} \quad (90)$$

which are the general forms of the real *convolution integral*, often also referred to as Borel's theorem. Actually, we recognize (74) and (82) as special forms, both illustrating the "folding over" of one of the time functions under the integral sign as demonstrated particularly in connection with the convolution summation (81) indicated in Fig. 1.16.

Table 1.5 shows several special forms of the convolution integral which are important in practical applications. To be able to apply these equally well for mesh and node-pair relations, it might be desirable to accept the term *immittance* (or *adpedance*) coined by Bode,^{B2} p. 15, and designated by $W(p)$ to represent either a parametric impedance $Z(p)$ or a parametric admittance $Y(p)$. Similarly, we might introduce a *general source function* $q(t)$ with transform $Q(p)$ which might represent either an applied voltage $v(t)$ or an applied current $i(t)$. The general form would thus read

$$W(p)\mathfrak{L}q(t) = W(p)Q(p) = \begin{cases} Z(p)I(p) \\ \text{or} \\ Y(p)V(p) \end{cases} \quad (91)$$

Referring to Table 1.4, we can now identify

$$\mathfrak{L}^{-1}W(p) = D(t) \quad (92)$$

as the unit impulse response, which for the lumped-parameter network can be expressed in terms of the expansion theorem given in line 11 of Table 1.4. Similarly we have

$$\mathfrak{L}^{-1}\frac{1}{p}W(p) = R(t) \quad (93)$$

the "indicial immittance" or indicial response function, or also unit step response. With these elements, the use of Table 1.5 becomes self-explanatory.

PROBLEMS

1.1 Compute the locus of the complex Fourier transform $\tilde{F}(\omega)$ for a single inverted sawtooth, i.e., having the sharp rise at the leading edge and decreasing to zero linearly within time T . Compare this locus with Fig. 1.7. Compute the amplitude and phase spectrum functions.

1.2 Compute the amplitude and phase spectrum functions for a train of three consecutive pulses of the shape described in problem 1.1. Describe the most significant differences between the spectrum functions for one and for three pulses (see discussions in section 6.4 of Vol. I).

1.3 Compare the spectrum functions for the single inverted sawtooth of problem 1.1 with those of the single pulse with $m = 1$ in section 1.2. Describe the most significant differences.

1.4 One pulse train consists of three like sawtooth pulses equally spaced as in Fig. 1.17*b*. Compare the amplitude and phase spectrum functions of these two pulse trains.

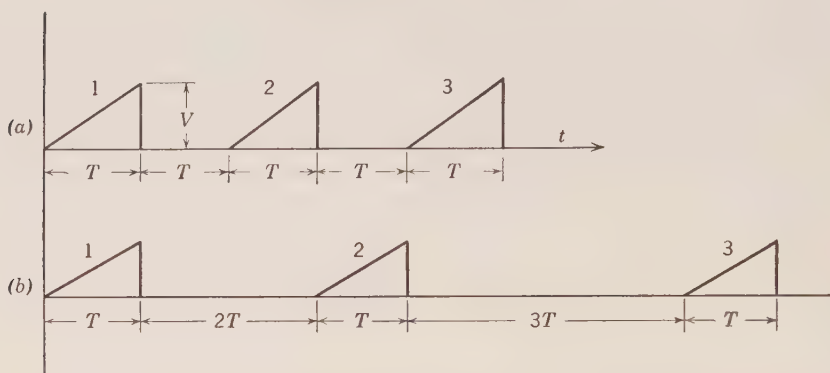


Fig. 1.17. Pulse trains of three like sawtooth pulses, (a) equally spaced, (b) spaced with increasing intervals.

1.5 Develop the amplitude and phase spectrum functions of a single pulse consisting of a half sine wave $\sin 2\pi t/T$ with $0 < t < T/2$. Compare these with the spectrum functions in Figs. 1.5 and 1.9.

1.6 A single sawtooth pulse such as $m = 0$ in Fig. 1.6 is sampled by rectangular pulses of the type shown in Fig. 1.14. Assume the sampling pulse width as $T/5n$ and the spacing as T/n . (a) Develop the analytical description of the sampling pulse sequence. (b) Develop the complex Fourier transform of the sampling pulse sequence and compare with that of the single continuous sawtooth pulse. (c) For numerical computations choose $n = 10$, divide T into 50 parts, and set the sampling pulse at the beginning of each fifth division following this pulse.

1.7 Assuming the sampling pulses in problem 1.6 to be equivalent to current impulse functions, apply the sequence to a parallel R, C circuit and find the output voltage. Compare this response with that of the same R, C circuit to the continuous single sawtooth current pulse represented by the sampling sequence.

1.8 The sine wave $A \sin \omega t$ starting at $t = 0$ is to be represented by sampling pulses which can be considered as impulse functions. Develop the complex Fourier transform of the sine wave as well as of a periodic sequence of sampling pulses and determine the smallest number of pulses per period needed to represent the sine wave in significant detail.

1.9 The current $i_0(t) = I_m \sin(\omega t + \psi)$ is applied at $t = 0$ to the parallel circuit shown in Fig. 1.12*c*. Find the output voltage $v_o(t)$. Check the steady-state values by the direct complex solution for the steady state.

1.10 The voltage impulse $M_o S_0(t)$ is applied to a series R, L, C circuit. Determine the voltage drop across capacitor C for $t \geq 0$.

1.11 The circuit in Fig. 1.18 has a step voltage $v_1(t) = V1$ applied at $t = 0$. Find the voltage $v_2(t)$ first with $r = 0$ and then correct for the small but finite value of the coil resistance r . Plot the response for $(LC)^{-1} = 10^{12} \text{ sec}^{-1}$, and $r = 0.01(L/C)^{1/2}$.

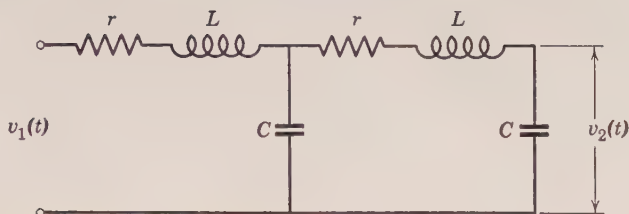


Fig. 1.18. Circuit for problem 1.11.

1.12 An impulse voltage $v_1(t) = M_v S_0(t)$ is applied to the circuit of Fig. 1.19. Find the output voltage $v_2(t)$. Find the output voltage $v_2(t)$ if $v_1(t)$ consists of a periodic sequence of impulse functions with spacing T .

1.13 A linear sawtooth current pulse of duration T (as in Fig. 1.6) is applied to the parallel L, G circuit Fig. 1.12c. What should be the value of T to get the highest maximum value of output voltage? For numerical computation choose $LG = 10^{-3}$ sec.

1.14 A voltage represented by the first quarter period of $V_m \cos 2\pi t/T$ is applied to the circuit in Fig. 1.19. Find the output voltage $v_2(t)$.

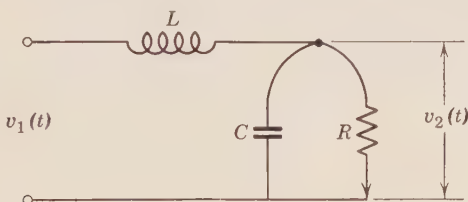


Fig. 1.19. Circuit for problems 1.12, 1.14, and 1.16.

1.15 Take the parallel G, L circuit of Fig. 1.12c. What must be the time variation of the applied current to give an output voltage that varies linearly with time?

1.16 A voltage pulse train of the form of Fig. 1.17a is applied at $t = 0$ to the circuit in Fig. 1.19. How does the inductance L affect the features of these pulses in the output voltage?

1.17 To a critically damped series R, L, C circuit we apply the three voltage pulse shapes shown in Fig. 1.6. What will be the corresponding current pulse shapes? For numerical computation assume $\delta = R/2L = 10^4 \text{ sec}^{-1}$, $T\delta = 1$.

1.18 An infinite sequence of sawtooth current pulses with alternating spacing $2T$ and $3T$ as in Fig. 1.17 is applied to the parallel L, G circuit of Fig. 1.12c. Find the steady-state output voltage (see section 5.7 in Vol. I).

Two-Terminal-Pair Networks

(Fourpoles)

Communication systems are basically cascade chains of individual networks each one of which transmits a signal containing desirable information from its input terminal pair to its output terminal pair with certain shaping actions upon it. The chain is a cascade because, in the sequence, the output pair of terminals of one network can be identified as the input pair of terminals of the succeeding network; the shaping functions might be modulation, amplification, selection, correction, and detection. The individual networks might be very simple or quite complex depending upon the desirability of subdivision by functions or by network components.

If we restrict our scope to linear circuits, whether active or passive, we can treat the action of a particular network upon the signal as independent of all other networks and turn the analysis into a systematic study. The early recognition of this tremendous advantage has led to the establishment of a particular notation specially suitable for the cascade connection, and to the introduction of the special designation *fourpole** as a "shorthand" notation, as it were, for the more cumbersome "two-terminal-pair network." It is important to accept the shorter expression only in the restricted sense of the defining long expression and not in its possible general meaning; this applies as well to the term four-terminal network.

Though the designation fourpole was first introduced into communication network theory, there is no reason to restrict it to this field; it is equally applicable to electric power transmission systems employing two wires, as well as to any analogous system in other fields of engineering. Frequently, fourpoles involving energy conversion from one form to another are designated as *transducers*† such as electromechani-

* The designation "Vierpol" (fourpole) was first used by F. Breisig, "On Cross-talk in Communication Circuits" [German], *Elektrotech. Z.*, **42**, 933-939 (1921).

† "Standards on Transducers: Definitions of Terms," *Proc. I.R.E.*, **39**, 897-900 (August 1951).

cal, electroacoustical transducers, and others in Vol. I, section 3. We also find designations such as converters, transformers, inverters, etc., all of which are characterized by input and output terminals and will be subsumed in the class of fourpoles if input and output each comprise a terminal pair, not excluding the special case where one input and one output terminal are common.

With this general concept in mind, we shall first treat the simpler and basic types of fourpoles, then progress to the chains of uniform fourpoles, and finally take up approximation methods.

2. PASSIVE FOURPOLES

Inasmuch as a simple fourpole requires for its characterization only the designation of input and output terminal pairs, it is convenient to denote it symbolically as in Fig. 2.1 by a box to which two terminal pairs are attached, either one of which may be taken as input pair or output pair. This arrangement leaves completely open the internal structure and may apply equally well to networks composed of only passive linear elements, or composed of active and passive linear elements. In the latter case, we shall call the entire fourpole "active," in the former case "passive."

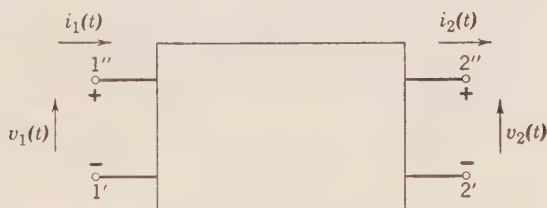


Fig. 2.1. Fourpole notation.

In this rather broad sense, we could apply this concept to almost any one of the examples treated in Vol. I as two-terminal networks, if the pertinent network can feasibly be inserted in a cascade chain. In turn, any lumped-parameter fourpole must admit the same methods of analysis as the two-terminal networks, so this section will not introduce new methods of transient analysis but rather will restrict itself to developing a systematic notation applicable to both passive and active linear fourpoles. We shall use the Laplace transform method throughout in the interest of uniformity and timeliness, and refer to other methods in cases where significant contributions have come from them.

2.1 Fourpole Notation

Taking cognizance of the fact that the concept of the fourpole has utility primarily for cascaded networks, the notation of input and output currents and voltages as shown in Fig. 2.1 is most suitable. At

the input terminals 1' and 1'', $v_1(t)$ is the applied voltage and $i_1(t)$ the impressed current, delivered by the preceding network sections. At the output terminals 2' and 2'', $v_2(t)$ should be counted positive from 2'' to 2' if considered as a passive voltage drop established across some terminating passive impedance; since it acts, however, as applied driving voltage at the input terminals of the succeeding fourpole, we count $v_2(t)$ positive in the direction 2' to 2'', which is a significant change compared with two-terminal networks, or *twopoles*. The output current should remain in the conventional direction shown in Fig. 2.1 so as to provide for the desired continuity in the cascade chain.*

Our primary interest is now to establish direct relationships between the input pair of quantities $v_1(t)$, $i_1(t)$ and the output pair $v_2(t)$, $i_2(t)$. Because of the assumed linearity of all elements, we must expect proportionality between the respective amplitudes that could also be



Fig. 2.2. Resistive T-pad.

deduced from the set of linear integro-differential equations which must relate these voltages and currents. To demonstrate this, we shall first examine the very simple case of a resistive T-network as shown in Fig. 2.2. The conventional network equations are here simply

$$\begin{aligned} (R_1 + R_m)i_1(t) - R_mi_2(t) &= v_1(t) \\ -R_mi_1(t) + (R_2 + R_m)i_2(t) &= -v_2(t) \end{aligned} \quad (1)$$

where the negative sign for v_2 in the second equation indicates a voltage counteracting the current $i_2(t)$ which has been assumed as positive. If we had a terminating resistance R_t so that $v_2(t) = R_ti_2(t)$, we see that this term could be brought to the left-hand side and there merged with R_2 as it should be. Because of the purely resistive character of the network, we can solve directly for any two quantities in terms of the remaining two. Selecting the input pair as desired, we get $i_1(t)$

* Several authors, notably Guillemin,^{A8} p. 134, and Cauer,^{A3} p. 45, have adopted the opposite direction of the output current $i_2(t)$, preferring the diagrammatic symmetry of the individual fourpole to the continuity within a cascade. As a matter of arbitrary convention rather than substance, this is not an important issue; see, however, the further discussion on p. 53.

from the second line, and introducing this into the first line we get $v_1(t)$, namely

$$\begin{aligned} v_1(t) &= \left(1 + \frac{R_1}{R_m}\right) v_2(t) + \left(R_1 + R_2 + \frac{R_1 R_2}{R_m}\right) i_2(t) \\ i_1(t) &= \frac{1}{R_m} v_2(t) + \left(1 + \frac{R_2}{R_m}\right) i_2(t) \end{aligned} \quad (2)$$

All the coefficients are constant, demonstrating the linear character of the system. We can bring this into matrix form, see Appendix 4,

$$\begin{bmatrix} v_1(t) \\ i_1(t) \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} v_2(t) \\ i_2(t) \end{bmatrix} \quad (3)$$

where α and δ are pure numbers, β is measured as a resistance and γ as a conductance, the whole group constituting the coefficient matrix of (2) which is the principal characteristic of the network.

Relations (2) can be used to deduce in a very simple manner the "principle of the equivalent generator" first given by Helmholtz* and later derived independently by Thévenin and now most frequently referred to as Thévenin's theorem.† Assume that a resistance R is connected across the output terminals 2' and 2'' and that we wish to find the current through it with the voltage $v_1(t)$ active at the input terminals of the network. If we set $v_2 = Ri_2$ in the first line of (2), we have the solution at once

$$i_2 = \frac{v_1(t)}{R[1 + (R_1/R_m)] + [R_1 + R_2 + (R_1 R_2)/R_m]}$$

For better interpretation we divide by $(1 + R_1/R_m)$ and write

$$i_2 = \frac{v_1'(t)}{R + \rho}, \quad v_1'(t) = \frac{v_1(t)}{1 + R_1/R_m} \quad (4)$$

where v_1' is the output voltage derived from the first line of (2) with $i_2 = 0$, i.e., open-circuited, and where ρ is the value $[v_2/(-i_2)]_{v_1=0}$ from the same relation. Thus, the current through the load resistance R can be found as if it were connected to a generator producing the actual open-circuit voltage v_1' across R and having an internal resistance ρ equal to the total network resistance seen from the terminals of

* H. Helmholtz, "On some laws of distribution of electric currents in solids . . ." [German], *Ann. Phys. Chemie*, Series 3, **29**, 222-224 (1853).

† L. Thévenin, "On a new electrodynamic theorem" (French), *Comptes Rendus*, **97**, 159 (1883). See also Letters to the Editor, Leon Charles Thévenin, *Elec. Eng.*, **69**, 186 (1950).

R with the source short-circuited. Though this theorem was first deduced for d-c flow as demonstrated in the foregoing, it is valid in the general a-c case as well since it is merely a matter of interpretation of a direct solution of a fourpole problem.

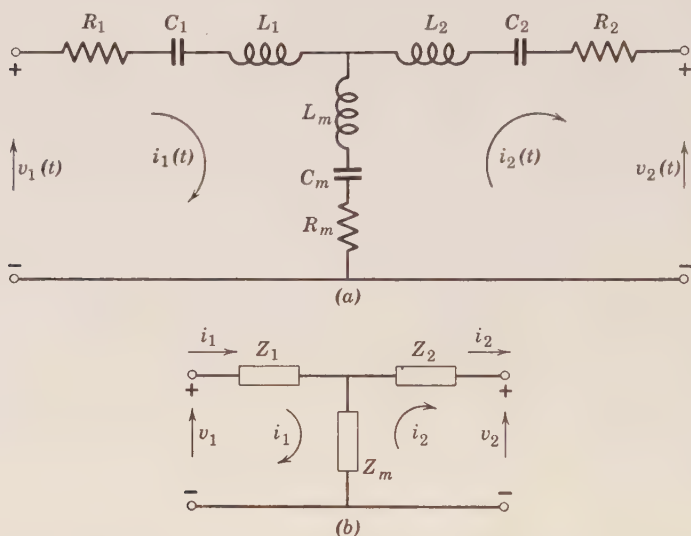


Fig. 2.3. General T-section: (a) specific form, (b) schematic form with parametric impedances.

As a more general illustration let us consider next the T-section shown in Fig. 2.3a. Each one of the network branches represents a total voltage drop

$$R_\alpha i_\beta(t) + L_\alpha \frac{di_\beta}{dt} + \frac{1}{C_\alpha} \int i_\beta dt \equiv B_\alpha(i_\beta) \quad \alpha, \beta = 1, 2, m \quad (5)$$

if we use an abbreviating designation as in Vol. I, (1.28). The conventional network equations are then quite analogous to (1),

$$\begin{aligned} B_1(i_1) + B_m(i_1) - B_m(i_2) &= v_1(t) \\ -B_m(i_1) + B_m(i_2) + B_2(i_2) &= -v_2(t) \end{aligned} \quad (6)$$

If we prefer to use double indices, we might write, as is frequently done, all coefficients with positive sign, so that

$$\begin{aligned} +B_{1,1}(i_1) + B_{1,2}(i_2) &= v_1(t) \\ +B_{2,1}(i_1) + B_{2,2}(i_2) &= -v_2(t) \end{aligned} \quad (7)$$

By comparison with (6)

$$B_{1,1} = B_1 + B_m, \quad B_{1,2} = B_{2,1} = -B_m, \quad B_{2,2} = B_2 + B_m$$

i.e., all the self terms are positive, all the mutual terms are actually negative; the negative sign of v_2 is specific for the fourpole convention.

Again we should like to reformulate (6) or (7) to express the input pair v_1 and i_1 in terms of the output pair v_2, i_2 . We can do this easily with complex notation for the a-c steady state, and with any of the operational or transform methods for the transient state, but we could not do this directly with the conventional integro-differential equations! It is this much greater facility of economic expression which has so greatly contributed to the rapid acceptance of the transform concept.

Let us apply the Laplace transformation to (6), denoting $\mathcal{L}v(t) = V(p)$ and $\mathcal{L}i(t) = I(p)$, each with the appropriate indices, and consult Table 1.4 for the derivative and integral transforms. We obtain for the general form (5)

$$\begin{aligned} \mathcal{L}B_\alpha(i_\beta) &= \left(R_\alpha + pL_\alpha + \frac{1}{pC_\alpha} \right) I_\beta(p) - L_\alpha i_\beta(0^+) + \frac{1}{pC_\alpha} q_\alpha^{(\beta)}(0^+) \\ &= Z_\alpha(p) I_\beta(p) + \left(-L_\alpha i_\beta(0^+) + \frac{1}{pC_\alpha} q_\alpha^{(\beta)}(0^+) \right) \end{aligned} \quad (8)$$

where $Z_\alpha(p)$ is the parametric impedance function for the branch α , $i_\beta(0^+)$ is the initial current, and $q_\alpha^{(\beta)}(0^+)$ that part of the initial charge on C_α attributable to the current i_β (which need not be known generally, because the network equations always contain the total current through C_α and therefore require only the total initial charge $q_\alpha(0^+)$ on C_α). The Laplace transform of the system (6) is thus

$$\begin{aligned} (Z_1 + Z_m)I_1(p) - Z_m I_2(p) &= V_1(p) + M_1(p) \\ -Z_m I_1(p) + (Z_2 + Z_m)I_2(p) &= -V_2(p) + M_2(p) \end{aligned} \quad (9)$$

which is easily visualized from the schematic diagram of Fig. 2.3b. The terms $M_\alpha(p)$ on the right-hand sides contain all the initial charge and current values, e.g.

$$M_1(p) = (L_1 + L_m)i_1(0^+) - L_m i_2(0^+) - \frac{1}{pC_1} q_1(0^+) - \frac{1}{pC_m} q_m(0^+)$$

where $q_m(0^+)$ is the total initial charge on C_m , taken positive if it supports the assumed positive direction of the current $i_1(t)$. These subsidiary terms can be read directly from the left-hand sides of (9) in

direct correspondence to the occurrence of derivative and integral transforms. It seems obvious that the extra terms $M_\alpha(p)$ are rather disturbing from the point of view of symmetry and elegance. We shall suppress them from now on, but must remind ourselves that this is tantamount to assuming an initially de-energized state. In a general solution of transient problems we must be cognizant of the existence of these terms and restore them at least by superposition of "extra currents" which express the decay of initial energies present in the system.

If we leave out the extra terms, (9) reduces in appearance to the same form as (1), with the parametric impedances taking the places of the resistances and with the respective Laplace transforms taking the places of the time functions of currents and voltages. We can therefore write the equivalent forms to (2) by inspection,

$$\begin{aligned} V_1(p) &= \left(1 + \frac{Z_1}{Z_m}\right) V_2(p) + \left(Z_1 + Z_2 + \frac{Z_1 Z_2}{Z_m}\right) I_2(p) \\ I_1(p) &= \frac{1}{Z_m} V_2(p) + \left(1 + \frac{Z_2}{Z_m}\right) I_2(p) \end{aligned} \quad (10)$$

All the coefficients are fractions of positive polynomials in p with real coefficients as long as we deal with passive fourpoles. We can consider (10) a *standard form* of fourpole relations, which can be expressed more generally as

$$\begin{aligned} V_1(p) &= \mathfrak{A}V_2(p) + \mathfrak{B}I_2(p) \\ I_1(p) &= \mathfrak{C}V_2(p) + \mathfrak{D}I_2(p) \end{aligned} \quad (11)$$

where \mathfrak{A} and \mathfrak{D} are dimensionless, \mathfrak{B} has the dimension of impedance, and \mathfrak{C} that of admittance. This standard notation can be generalized to all linear fourpoles, whether passive or active and has considerable advantage for the systematic treatment of fourpole theory. In matrix form, (11) reads simply

$$\begin{bmatrix} V_1(p) \\ I_1(p) \end{bmatrix} = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix} \begin{bmatrix} V_2(p) \\ I_2(p) \end{bmatrix} \quad (12)$$

where by comparison of (11) with (10) for the generalized passive T-section* of Fig. 2.3*b*

* A general comparison of fourpole notation is given by W. R. MacLean, "A Compilation of Transducer Formulae," *Communications*, **24**, 58–66 (Nov. 1944); see also Guillemin,^{A8} Vol. II, pp. 134–138, Cauer,^{A9} p. 124.

$$\begin{aligned}\mathfrak{A} &= 1 + \frac{Z_1}{Z_m}, & \mathfrak{B} &= Z_1 + Z_2 + \frac{Z_1 Z_2}{Z_m} \\ \mathfrak{C} &= \frac{1}{Z_m}, & \mathfrak{D} &= 1 + \frac{Z_2}{Z_m}\end{aligned}\quad (13)$$

It is rather obvious that (12) is merely a reiteration of the principle of superposition valid for linear systems; it is contained in Kirchhoff's early work and stressed by Maxwell.*

Actually, the notation in (11) was first introduced in the theory of long power transmission lines† and is there used invariably as in (11) and with the directions of currents as in Fig. 2.1; see Dahl,^{C2} chapter IX, Guillemin,^{C5} Vol. II, p. 69, Woodruff,^{C12} chapter VI, Sah,^{C10} p. 230, Skilling,^{C11} p. 274. In communication lines and networks, form (11) seems to have been used first by Breisig,‡ whereas the application of the matrix notation is credited to Strecker and Feldtkeller.§ The parameters \mathfrak{A} , \mathfrak{B} , \mathfrak{C} , \mathfrak{D} are designated as *general circuit constants* if they pertain to a single frequency as in power systems, and *general fourpole parameters* in communication networks where they are normally interpreted as functions of frequency. As pointed out previously, several authors prefer the symmetrical designation of current flow in fourpoles, notably Guillemin,^{A8} Vol. II, p. 134, and many of his pupils, and Cauer.^{A3} To preserve the standard definition of the \mathfrak{A} , \mathfrak{B} , \mathfrak{C} , \mathfrak{D} parameters, we find then the current transform $I_2(p)$ with negative sign. In comparing relations found in the literature, it is therefore important to check which convention has been followed and to be particularly careful with signs.

The basic linear relationships can also be systematized in terms of an impedance matrix. If we return to (9) we observe that attaching a negative sign to the current i_2 will make all terms positive (disregarding again the $M(p)$ terms), so that we obtain in systematic notation with double index impedances,

$$\begin{aligned}Z_{11}(p)I_1(p) + Z_{12}(p)[-I_2(p)] &= V_1(p) \\ Z_{21}(p)I_1(p) + Z_{22}(p)[-I_2(p)] &= V_2(p)\end{aligned}\quad (14)$$

* C. Maxwell, *A Treatise on Electricity and Magnetism*, 3rd edition, Vol. I, pp. 280–283, Clarendon Press, Oxford, 1873.

† R. D. Evans and H. K. Sels, "Transmission line constants and resonance," *Elec. J.*, **18**, 306 (July 1921).

‡ F. Breisig, *Theoretische Telegraphie* [*Theoretical Telegraphy*], F. Vieweg and Sohn, Braunschweig, 2nd edition, pp. 346, 352, and 378, 1924.

§ F. Strecker and R. Feldtkeller, "Grundlagen der Theorie des allgemeinen Vierpols" ["Fundamentals of General Fourpole Theory"], *Elek. Nachr. Tech.*, **6**, 93–112 (1929).

For the individual fourpole, the symmetrical choice of current flow is therefore advantageous *if* we choose an impedance (or admittance) representation. If we solve (14) for the currents, we obtain the equally symmetrical form

$$\begin{aligned} Y_{11}(p)V_1(p) + Y_{12}V_2(p) &= I_1(p) \\ Y_{21}(p)V_1(p) + Y_{22}V_2(p) &= [-I_2(p)] \end{aligned} \quad (15)$$

where

$$\begin{aligned} Y_{11} &= \frac{Z_{22}}{|Z|}, & Y_{12} &= \frac{-Z_{12}}{|Z|}, & Y_{21} &= \frac{-Z_{21}}{|Z|}, & Y_{22} &= \frac{Z_{22}}{|Z|} \\ |Z| &= Z_{11}Z_{22} - Z_{12}Z_{21} \end{aligned} \quad (16)$$

and conversely for the impedances in terms of the admittances.

It is now comparatively easy to attach direct physical significance to the parameters, either appearing in (12) as \mathfrak{A} , \mathfrak{B} , \mathfrak{C} , \mathfrak{D} , in (14) as $Z_{\alpha\beta}(p)$, or in (15) as $Y_{\alpha\beta}(p)$, if we produce open-circuit or short-circuit conditions and read from the equations the ratios of terms as indicated:

For $I_2 = 0$

$$\begin{aligned} \mathfrak{A} &= \left. \frac{V_1(p)}{V_2(p)} \right|_{I_2=0} = \frac{Z_{11}}{Z_{21}} = -\frac{Y_{22}}{Y_{21}} && \text{open-circuit} \\ &&& \text{transfer function} \\ \frac{1}{\mathfrak{C}} &= \left. \frac{V_2(p)}{I_1(p)} \right|_{I_2=0} = Z_{21} && \text{open-circuit} \\ &&& \text{transfer impedance} \\ \frac{\mathfrak{A}}{\mathfrak{C}} &= \left. \frac{V_1(p)}{I_1(p)} \right|_{I_2=0} = Z_{11} && \text{open-circuit} \\ &&& \text{driving-point impedance} \end{aligned} \quad (17)$$

For $V_2 = 0$

$$\begin{aligned} \frac{1}{\mathfrak{B}} &= \left. \frac{I_2(p)}{V_1(p)} \right|_{V_2=0} = -Y_{21} && \text{short-circuit} \\ &&& \text{transfer admittance} \\ \mathfrak{D} &= \left. \frac{I_1(p)}{I_2(p)} \right|_{V_2=0} = \frac{Z_{22}}{Z_{21}} = -\frac{Y_{11}}{Y_{21}} && \text{short-circuit} \\ &&& \text{transfer function} \\ \frac{\mathfrak{D}}{\mathfrak{B}} &= \left. \frac{I_1(p)}{V_1(p)} \right|_{V_2=0} = Y_{11} && \text{short-circuit} \\ &&& \text{driving-point admittance} \end{aligned} \quad (18)$$

Similar relations are, of course, possible for the two additional conditions $I_1 = 0$ and $V_1 = 0$, respectively. Because of the attachment of direction signs to the currents, the definitions of, and interrelations

between, the parameters are identical* with those in Guillemin,^{A8} Vol. II, p. 138, and Cauer,^{A3} p. 124.

Definitions (17) and (18) also outline the modes of measurement of the parameters. In the most general case, four measurements are needed in order to give the values of the four parameters; this will be true for nonreciprocal passive and for many active networks, as well as for transducers involving different kinds of energy, such as electro-mechanical systems.

2.2 Basic Types of Passive Fourpoles

Considering a *passive fourpole* consisting only of *bilateral, linear passive* elements, then *reciprocity* must exist, i.e., we must have

$$Y_{21} = Y_{12}, \quad Z_{21} = Z_{12}$$

The parameter determinant, when evaluated with relations (17) and (18) as well as (16), has the value

$$\begin{vmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{vmatrix} = \mathfrak{A}\mathfrak{D} - \mathfrak{B}\mathfrak{C} = \frac{Z_{11}Z_{22}}{Z_{21}^2} + \frac{1}{Z_{21}Y_{21}} = 1 \quad (19)$$

so that only three parameters are necessary, the fourth one being defined by (19).

A further restriction might be imposed by requiring that the passive fourpole be also symmetrical, i.e., that it should not matter which pair of terminals be designated as input and which as output. The *conditions of symmetry* can be deduced by inverting the system (11) and requiring identity of the result with (11). Solving for V_2 , I_2 and observing (19), we obtain

$$\begin{aligned} V_2(p) &= \mathfrak{D}V_1(p) - \mathfrak{B}I_1(p) \\ I_2(p) &= -\mathfrak{C}V_1(p) + \mathfrak{A}I_1(p) \end{aligned} \quad (20a)$$

If we now also reverse the currents in keeping with the reversal of the fourpole (making V_2 the active input voltage), then

$$\begin{aligned} V_2(p) &= \mathfrak{D}V_1(p) + \mathfrak{B}I_1(p) \\ I_2(p) &= \mathfrak{C}V_1(p) + \mathfrak{A}I_1(p) \end{aligned} \quad (20b)$$

The reversal of the fourpole interchanges only the two parameters \mathfrak{A} and \mathfrak{D} which sometimes are therefore designated as *external*, to differentiate from \mathfrak{B} and \mathfrak{C} which remain unchanged as *internal* parameters.

* We have kept Y_{21} and Z_{21} as different from Y_{12} and Z_{12} , respectively, for greater generality whereas the references assume these as equal.

The *condition of symmetry* for a fourpole is thus $\mathfrak{A} \equiv \mathfrak{D}$; the symmetrical fourpole only needs *two independent parameters* to be completely determined.

If a passive fourpole contains one or more nonbilateral elements, then (19) will not be true; we obtain in such cases

$$\begin{vmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{vmatrix} = \frac{Z_{12}}{Z_{21}} = \frac{Y_{12}}{Y_{21}} \quad (19a)$$

The value of the determinant of the general fourpole parameters is therefore a direct indication of the presence of a nonbilateral linear element. The most important special case obtains for

$$Z_{12} = -Z_{21}, \quad Y_{12} = -Y_{21}$$

which is generally associated with gyrator action. The concept of the gyrator was defined by Tellegen* in terms of the voltage-current relations

$$V_1 = -\alpha I_2$$

$$V_2 = \alpha I_1$$

with α real and with the significance of a resistance. An example was treated in Vol. I, Fig. 3.10, representing the moving-coil microphone. More extensive treatments are given by Tellegen† and Carlin.‡

It is quite obvious that for the gyrator network

$$\mathfrak{A}\mathfrak{B} - \mathfrak{C}\mathfrak{D} = -1 \quad (19b)$$

as contrasted with (19) for the reciprocal network. However, only three parameters are necessary to be obtained by independent measurements.

Since one of the principal advantages of the *general parameter notation* is its easy application to the cascading of simpler fourpoles into more complicated structures or chains, we might define the \mathfrak{A} , \mathfrak{B} , \mathfrak{C} , \mathfrak{D} values for individual branches and combine these elementary fourpoles into the basic types of fourpole structures.

Suppose we start with a single series impedance $Z(p)$. It does not matter whether we arrange Z in one conductor or whether we place $Z/2$ in each conductor connecting the input and output terminals as in

* B. D. H. Tellegen, "The Gyrator—A New Electric Network Element," *Philips Research Repts.*, **3**, 81–101 (1948).

† B. D. H. Tellegen and E. Klaus, "The Parameters of a Passive Four-Pole that may Violate the Reciprocity Relation," *Philips Research Repts.*, **5**, 81–86 (1950).

‡ H. J. Carlin, "On the Physical Realizability of Nonreciprocal Networks," *Proc. I.R.E.*, **43**, 608–616 (1955).

Fig. 2.4a and b. The simpler arrangement is Fig. 2.4a which could also be designated a three-terminal network though we shall refer to it as fourpole. We can determine the standard parameters by the operations indicated in (17) and (18). Leaving $2'$ and $2''$ isolated, we have from the figure directly for the currents and voltages, and therefore their Laplace transforms, $I_1 = I_2 = 0$, $V_1 = V_2$; thus $\alpha = 1$, $c = 0$. If we short-circuit terminals $2'$ and $2''$ we have $V_1 = ZI_1$,

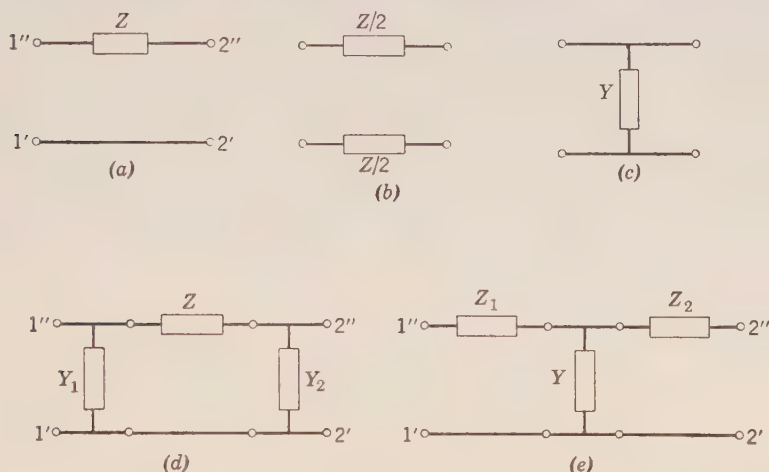


Fig. 2.4. Elementary fourpoles: (a) and (b) series impedance, (c) shunt admittance, (d) basic II-section, (e) basic T-section.

$I_1 = I_2$, $V_2 = 0$; thus $\beta = Z$, $\mathfrak{D} = 1$. The series impedance $Z(p)$ is thus defined by

$$\boxed{Z(p)} = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \quad (21)$$

satisfying (19) as well as the symmetry condition $\alpha = \mathfrak{D}$. For the single shunt admittance $Y(p)$ in Fig. 2.4c we derive in similar manner

$$\boxed{Y(p)} = \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \quad (22)$$

With these elements we can now synthesize some of the basic fourpole structures. Obviously, series connection of series impedances or shunt connection of shunt admittances does not lead to new results, since we can write from elementary considerations simply the sum of the elements into the pertinent matrix space. New structures can be achieved only by *alternation* of these elementary fourpoles, leading to the two basic forms: the II-section, Fig. 2.4d, and the T-section, Fig.

2.4e. By matrix multiplication (see Appendix 4) we find for the Π -section

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{\Pi} = \begin{bmatrix} 1 & 0 \\ Y_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_2 & 1 \end{bmatrix} \\ = \begin{bmatrix} 1 + Y_2 Z & Z \\ Y_1 + Y_2 + Y_1 Y_2 Z & 1 + Y_1 Z \end{bmatrix} \quad (23)$$

If $Y_1 = Y_2$ this reduces to the symmetrical Π for which we choose the self-explanatory notation

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{A} \end{bmatrix}_{\Pi} = \begin{bmatrix} 1 + YZ & Z \\ Y(2 + YZ) & 1 + YZ \end{bmatrix} \quad (24)$$

For the T-section we get by the same method the same values as in (13), if we replace there Z_m by Y^{-1} , namely

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{\text{T}} = \begin{bmatrix} 1 + YZ_1 & Z_1 + Z_2 + Z_1 Z_2 Y \\ Y & 1 + YZ_2 \end{bmatrix} \quad (25)$$

Again, if $Z_1 = Z_2$ this reduces to the symmetrical T for which we write

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{A} \end{bmatrix}_{\text{T}} = \begin{bmatrix} 1 + YZ & Z(2 + YZ) \\ Y & 1 + YZ \end{bmatrix} \quad (26)$$

Comparison between the general forms (23) and (25) shows that replacement of Y_1, Y_2, Z respectively by Z_2, Z_1, Y makes the elements

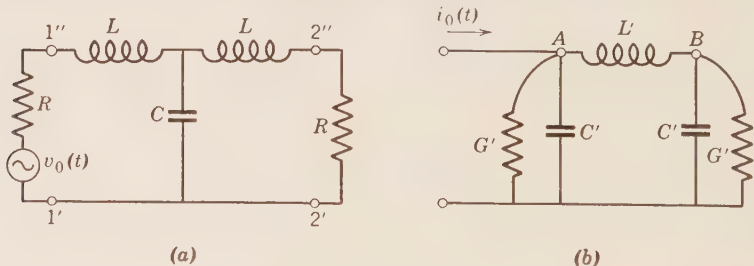


Fig. 2.5. The T-section (a) is the dual to the Π -section (b).

in the two matrices identical, except that \mathfrak{B} and \mathfrak{C} are interchanged. This indicates a dual nature which is borne out if we apply the construction outlined in Vol. I, section 3.2, to either structure of Fig. 2.4d or e. For the simple T-section in Fig. 2.5a, we find the dual as the Π -section in Fig. 2.5b with

$$\frac{R}{G'} = \frac{L}{C'} = \frac{L'}{C} = r^2$$

and with the voltage source converted into the equivalent current source. The transient solution for both structures must then be identical.

A frequently used modified structure is the *bridged T* in Fig. 2.6a. This might serve as a convenient example of determining the *general parameters* by direct computation, utilizing the defining ratios from (17) and (18). For the open-circuit condition $I_2 = 0$, the network

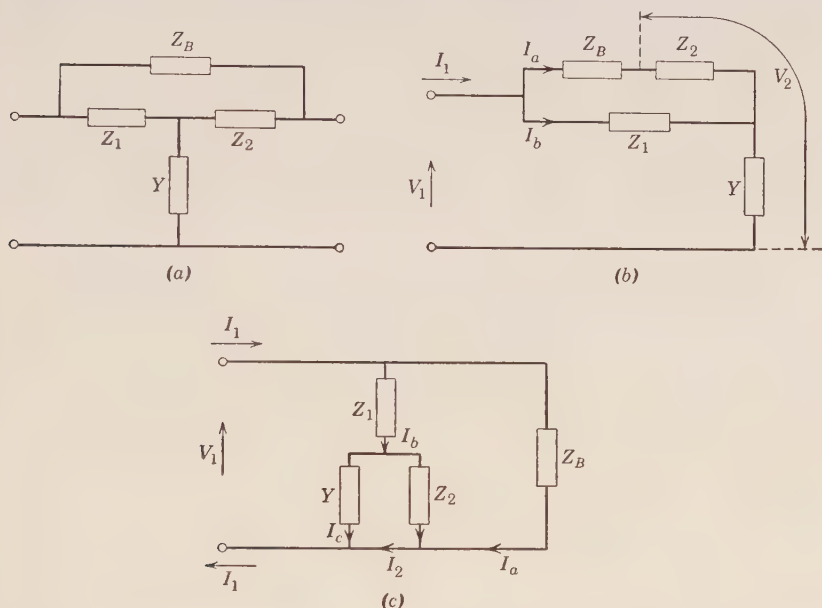


Fig. 2.6. Bridged-T structure (a); redrawn in (b) for open-circuit computation and in (c) for short-circuit computation.

has been redrawn in Fig. 2.6b. The total input impedance is by inspection

$$Z_{11} = \frac{1}{Y} + \frac{Z_1(Z_2 + Z_B)}{Z_1 + Z_2 + Z_B} = \frac{\Sigma + YZ_1(Z_2 + Z_B)}{Y\Sigma} \quad (27)$$

where

$$\Sigma = Z_1 + Z_2 + Z_B \quad (28)$$

is the sum of all impedances. We compute now the voltage V_2 as marked in the figure,

$$V_2 = \frac{I_1}{Y} + I_a Z_2 = \left(\frac{1}{Y} + \frac{Z_1}{\Sigma} Z_2 \right) I_1 = Z_{21} I_1$$

I_a was determined from the conditions

$$I_a(Z_B + Z_2) = I_b Z_1, \quad I_a + I_b = I_1$$

We thus obtain from (17) for the bridged T, using the subscript BT,

$$\begin{aligned} \mathfrak{C}_{BT} &= \frac{1}{Z_{21}} = Y \frac{\Sigma}{\Sigma + YZ_1 Z_2} \\ \mathfrak{G}_{BT} &= \frac{Z_{11}}{Z_{21}} = 1 + YZ_1 \frac{Z_B}{\Sigma + YZ_1 Z_2} \end{aligned} \quad (29)$$

In both instances we note that if $Z_B \rightarrow \infty$ the fractions approach unity, so that the parameters reduce smoothly to the values for the plain T given in (25). For the short-circuit condition $V_2 = 0$, the network has been redrawn in Fig. 2.6c. The total input admittance follows from the figure by inspection

$$Y_{11} = \frac{1}{Z_B} + \frac{1 + YZ_2}{Z_1 + Z_2 + YZ_1 Z_2} = \frac{\Sigma + YZ_2(Z_1 + Z_B)}{Z_B(Z_1 + Z_2 + YZ_1 Z_2)}$$

We note that the denominator contains \mathfrak{G}_T , the parameter for the plain T section. The current I_2 can be found from the figure

$$I_2 = \left(\frac{1}{Z_B} + \frac{1}{Z_1 + Z_2 + YZ_1 Z_2} \right) V_1$$

so that

$$\begin{aligned} \mathfrak{B}_{BT} &= \frac{I_2}{V_1} = (Z_1 + Z_2 + YZ_1 Z_2) \frac{Z_B}{\Sigma + YZ_1 Z_2} \\ \mathfrak{D}_{BT} &= Y_{11} \frac{V_1}{I_2} = 1 + YZ_2 \frac{Z_B}{\Sigma + YZ_1 Z_2} \end{aligned} \quad (30)$$

with Σ from (28). Again we note that as $Z_B \rightarrow \infty$ the fractions approach unity, and the parameters reduce to those of the plain T in (25). It is also apparent that the simple addition of the branch Z_B complicates the parameter expressions very considerably so that a complete transient analysis becomes rather cumbersome.

A somewhat different type of fourpole is the *bridge* shown in Fig. 2.7a, which has as its earliest prototype the *Wheatstone bridge* with four resistance arms, but which has been used in innumerable modifications for measurement purposes.* In the communication field, its equivalent, obtained by simple redrawing as in Fig. 2.7b, is known as the

* See the very comprehensive treatment by B. Hague, *Alternating Current Bridge Methods*, Pitman, London, 1923.

lattice section. Application of the same method as above leads directly to the general parameters of the lattice section

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}_L = \frac{1}{QS - TP} \begin{bmatrix} (P + Q)(S + T) & (Q + S)PT + (P + T)QS \\ (P + Q) + (S + T) & (P + S)(Q + T) \end{bmatrix} \quad (31)$$

The factor in front of the right-hand matrix applies to all terms of this matrix. If we now make $P = T$ and $Q = S$ so that the lattice section

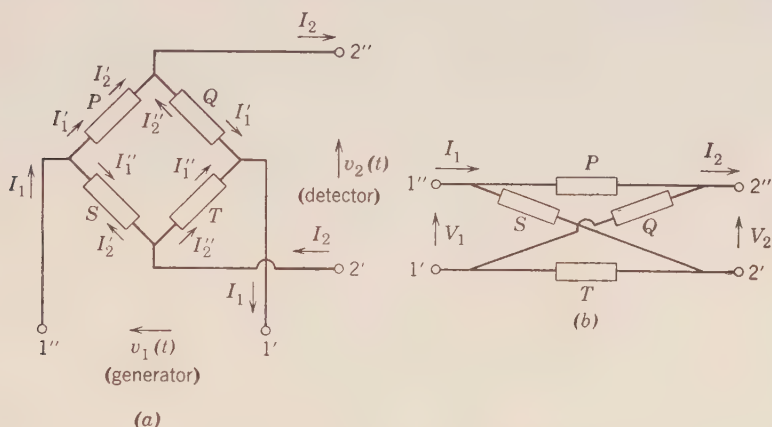


Fig. 2.7. Bridge (a) and its equivalent, the lattice section (b).

becomes *symmetrical*, we achieve considerable simplification, namely

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \alpha \end{bmatrix}_L = \frac{1}{Q - P} \begin{bmatrix} Q + P & 2QP \\ 2 & Q + P \end{bmatrix} \quad (32)$$

where again $(Q - P)^{-1}$ applies to all the terms.

If it is not possible to adjust the elements of an unsymmetrical fourpole to make it symmetrical, we can use the cascade connection of two identical unsymmetrical fourpoles "back to back." As shown in (20), reversal of the fourpole interchanges in its matrix only α and δ ; the combination will therefore have the following over-all parameters

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \times \begin{bmatrix} \delta & \beta \\ \gamma & \alpha \end{bmatrix} = \begin{bmatrix} 2\alpha\delta - 1 & 2\alpha\beta \\ 2\gamma\delta & 2\alpha\delta - 1 \end{bmatrix} \quad (33)$$

which clearly shows the desired symmetry.

Though we have stressed the cascade arrangement of fourpoles as the most important interconnection, two fourpoles can also be con-

nected in series, in parallel, or in mixed arrangements; see Guillemin,^{A8} Vol. II, pp. 145-151. Little seems gained by a general systematic treatment since most cases are more readily analyzed on an individual basis.

2.3 Transients in Simple Passive Fourpoles

The principal purpose of the separate study of transients in passive fourpoles is to demonstrate the high degree of generality which can be achieved with the general parameter notation \mathfrak{A} , \mathfrak{B} , \mathfrak{C} , \mathfrak{D} . Of course, specific solutions can be given only when the exact structure of the fourpole is known.

Let us assume a general fourpole indicated by the box in Fig. 2.8 and terminated at the input into impedance Z_{t1} and at the output

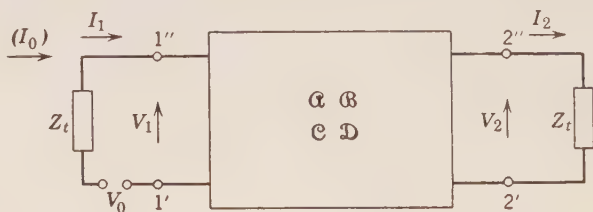


Fig. 2.8. General fourpole with balanced termination.

into the impedance Z_{t2} . The energy source might have an internal impedance (admittance) which is assumed to be contained in Z_{t1} . The terminal conditions are therefore, employing Laplace transforms throughout,

$$\left. \begin{array}{l} \text{Voltage source } V_0: \quad V_1 = V_0 - I_1 Z_{t1} \\ \text{Current source } I_0: \quad V_1 = (I_0 - I_1) Z_{t1} \end{array} \right\} V_2 = Z_{t2} I_2 \quad (34)$$

and we consider generally as the solution of our problem the explicit expression of I_2 in terms of the source function. If the output voltage V_2 is desired, the appropriate modifications are rather obvious.

With (34) we can write the basic fourpole relations (11) in the form

$$\begin{aligned} (I_0 Z_{t1} = V_0) &= Z_{t1} I_1 + (\mathfrak{A} Z_{t2} + \mathfrak{B}) I_2 \\ I_1 &= (\mathfrak{C} Z_{t2} + \mathfrak{D}) I_2 \end{aligned} \quad (35)$$

valid for either current source or voltage source by using the appropriate term from the parenthesis of the first line. Introducing the second line into the first, we obtain the solution

$$I_2 = (\mathfrak{C} Z_{t1} Z_{t2} + \mathfrak{A} Z_{t2} + \mathfrak{D} Z_{t1} + \mathfrak{B})^{-1} \left\{ \begin{array}{l} V_0 \\ Z_{t1} I_0 \end{array} \right. \quad (36)$$

Theoretically, simplification can be achieved if an ideal voltage source is applied directly to the terminals of the fourpole, i.e., if we choose $Z_{t1} = 0$,

$$I_{2(Z_{t1}=0)} = \frac{V_0}{\alpha Z_{t2} + \beta} \quad (37)$$

or if we apply an ideal current source directly to the terminals so that $Z_{t1} = \infty$,

$$I_{2(Z_{t1}=\infty)} = \frac{I_0}{\epsilon Z_{t2} + \mathfrak{D}} \quad (38)$$

An important group of networks comprises symmetrical fourpoles with balanced terminations (taking into account the internal source contribution) so that $\alpha = \mathfrak{D}$ and $Z_{t1} = Z_{t2} = Z_t$. If we also observe $\alpha^2 - \beta\epsilon = 1$ from (19), we can write for the parenthesis in (36)

$$\epsilon Z_t^2 + 2\alpha Z_t + \frac{\alpha^2 - 1}{\epsilon} = \frac{1}{\epsilon} [(\epsilon Z_t + \alpha)^2 - 1]$$

This permits easy factoring, so that the general solution takes the form

$$I_2 = \frac{\epsilon V_0}{(\epsilon Z_t + \alpha + 1)(\epsilon Z_t + \alpha - 1)} \quad (V_0 = I_0 Z_t) \quad (39)$$

where the parenthetical expression on the right indicates the simple substitution for a current source. Solution (39) can have a distinct advantage since it has factored the denominator irrespective of the specific structure of the fourpole and involves actually only two parameters, α and ϵ . An alternative form can be obtained, if we replace ϵ rather than β from the fourpole determinant, $\epsilon = (\alpha^2 - 1)/\beta$, so that

$$\frac{\alpha^2 - 1}{\beta} Z_t^2 + 2\alpha Z_t + \beta = \frac{1}{\beta} [(\alpha Z_t + \beta)^2 - Z_t^2]$$

This again permits factoring and leads to

$$I_2 = \frac{\beta V_0}{(\alpha Z_t + \beta + Z_t)(\alpha Z_t + \beta - Z_t)} \quad (V_0 = I_0 Z_t) \quad (40)$$

It will depend on the specific structure of the fourpole which of the three forms (36), (39), or (40) will be simplest to use in a given instance. For the plain T-section we see from (26) that (39) would be simpler, whereas for the Π -section (24) indicates (40) would be preferable.

As an illustration we shall compute the transient response to a suddenly applied sinusoidal voltage of the symmetrical T-section in

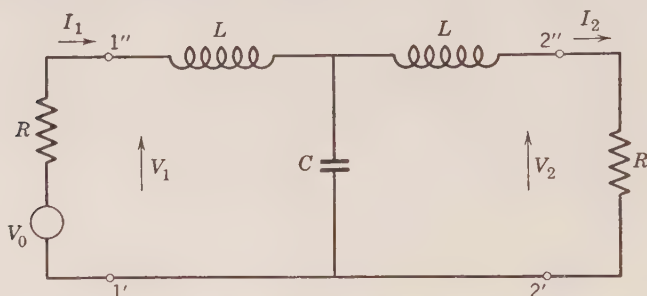


Fig. 2.9. Symmetrical low-pass T-section with balanced resistive terminations.

Fig. 2.9 with balanced resistive terminations. The voltage transform is in complex notation as seen in Table 1.3.,

$$\bar{V}_0(p) = \mathcal{L}(V_m e^{j\omega t}) = \frac{V_m}{p - j\omega}$$

and the elements of the structure are in parametric notation

$$Z = pL, \quad Y = pC, \quad Z_t = R$$

so that

$$\begin{aligned} \alpha = \mathfrak{D} &= 1 + YZ = 1 + p^2 LC, & \mathfrak{C} &= Y = pC \\ \mathfrak{B} &= Z(2 + YZ) = pL(2 + p^2 LC) \end{aligned} \quad (41)$$

This obviously invites the use of form (39) since \mathfrak{B} appears rather involved. We obtain for the denominator of (39)

$$(pCR + p^2 LC + 2)(pCR + p^2 LC) \quad (42)$$

rather conveniently factored for finding the root values. This same problem has been treated in Vol. I, section 5.3, Fig. 5.3. Actually, (5.49) from Vol. I demonstrates the same result for the denominator function but obtained from the conventional mesh equations. Surely, in this very simple case one might question the need for the α , \mathfrak{B} , \mathfrak{C} , \mathfrak{D} notation and the very general treatment leading to (39). However, (39) is true in this simple form for *any symmetrical structure* between the terminal pairs no matter how complicated the system of mesh equations will look!

To carry the solution through in simple form, let us introduce the following notation

$$\Omega_c = \frac{\sqrt{2}}{\sqrt{LC}}, \quad \frac{p}{\Omega_c} = y$$

and let us assume that the time constant

$$RC = T_0 = \frac{2}{\Omega_c}$$

or also that

$$\Omega_c T_0 = 2$$

We shall call Ω_c the *cutoff frequency* of the fourpole. We thus obtain from (39) in complex notation,

$$\bar{I}_2(p) = \frac{\Omega_c C V_m}{4} \frac{1}{(p - j\omega)(y + 1)(y^2 + y + 1)} \quad (43)$$

The root values of the denominator are, respectively,

$$\begin{aligned} y_2 &= -1, & y_{3,4} &= -\frac{1}{2}(1 \mp j\sqrt{3}) \\ p_1 &= j\omega, & p_2 &= -\Omega_c, & p_{3,4} &= -\frac{\Omega_c}{2}(1 \mp j\sqrt{3}) \end{aligned} \quad (44)$$

Restricting ourselves to the characteristic fraction in (43), the residues at the four simple poles are, respectively, if we also introduce $\mu = \omega/\Omega_c$ for the frequency ratio,

$$\begin{aligned} &\frac{e^{j\omega t}}{(1 + j\mu)(1 + j\mu - \mu^2)}, & \frac{\Omega_c}{(-\Omega_c - j\omega)} e^{-\Omega_c t}, \\ &\frac{4\Omega_c}{[-\Omega_c(1 \mp j\sqrt{3}) - 2j\omega](1 \pm j\sqrt{3})(\pm j\sqrt{3})} e^{-\frac{\Omega_c}{2}(1 \mp j\sqrt{3})t} \end{aligned}$$

Rationalizing and taking the imaginary part of these to obtain the inverse Laplace transform of (43), we arrive at

$$\begin{aligned} i_2(t) &= \frac{V_m}{2R} \left\{ \frac{\sin \{ \omega t - \tan^{-1} \mu - \tan^{-1} [\mu/(1 - \mu^2)] \}}{(1 + \mu^2)^{1/2} [(1 - \mu^2)^2 + \mu^2]^{1/2}} + \frac{\mu}{1 + \mu^2} e^{-\Omega_c t} \right. \\ &\quad \left. + \frac{\mu}{(1 + \mu^2)^2 - 3\mu^2} e^{-\frac{\Omega_c}{2} t} \right. \\ &\quad \left. \left[(1 - \mu^2) \cos \left(\frac{\sqrt{3}}{2} \Omega_c t \right) + \frac{1 + \mu^2}{\sqrt{3}} \sin \left(\frac{\sqrt{3}}{2} \Omega_c t \right) \right] \right\} \quad (45) \end{aligned}$$

Comparison with the solution for an applied step voltage given in Vol. I, p. 178, shows the same transient modes but different amplitudes as we must expect.

Suppose we examine the steady-state response as a function of applied frequency ω and of terminating resistance R , keeping the net-

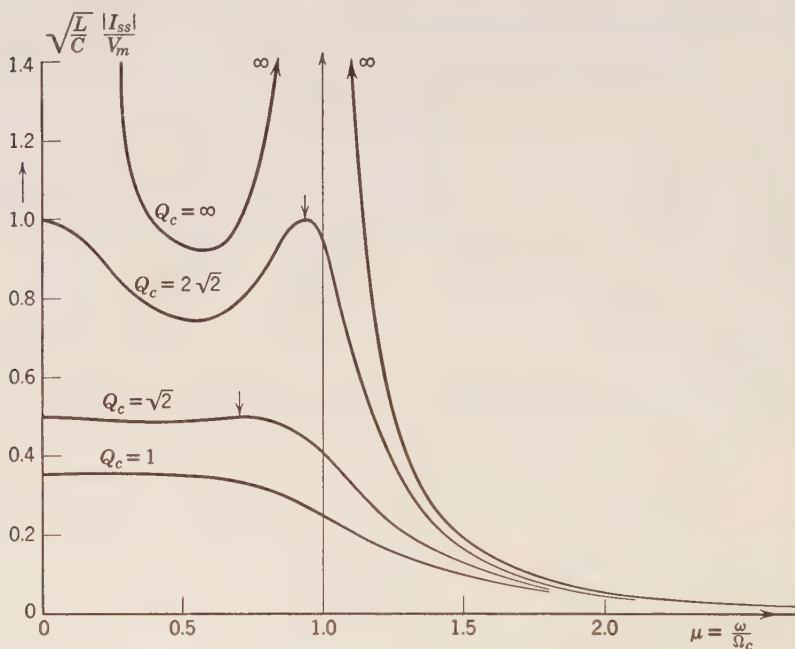


Fig. 2.10. Steady-state current response of low-pass T-section with balanced resistive terminations.

work parameters L and C constant. We shall consequently use $\mu = \omega/\Omega_c$ as variable and select as parameter

$$Q_c = \frac{\Omega_c L}{R} = \frac{\sqrt{2}}{R} \sqrt{\frac{L}{C}}$$

Returning to (39) we get the amplitude of the current response by replacing the current and voltage transforms by the respective phasors $I_2(\omega)$ and $V(\omega) = V_m$ (we had taken zero phase angle of the applied voltage), and letting $p \rightarrow j\omega$. The absolute value becomes

$$|I_2(\omega)| = |I_{ss}| = \frac{V_m}{2R} \frac{1}{Q_c} \left[(1 - \mu^2)^2 + \left(\frac{\mu}{Q_c} \right)^2 \right]^{-1/2} \left(\mu^2 + \frac{1}{Q_c^2} \right)^{-1/2} \quad (46)$$

This agrees with the amplitude of the steady-state term in (45) if we

put $Q_c = 1$ in (46). Fig. 2.10 shows the plot of $|I_{ss}|$ against μ with the parameter values $Q_c = 1, \sqrt{2}, 2\sqrt{2}, \infty$. We can readily find the maximum and minimum values by differentiating the radicand in (46) with respect to μ ,

$$\begin{aligned} \frac{d}{d\mu} \left\{ \left[(1 - \mu^2)^2 + \left(\frac{\mu}{Q_c} \right)^2 \right] \left(\mu^2 + \frac{1}{Q_c^2} \right) \right\} \\ = 2\mu \left[3\mu^4 + 4\mu^2 \left(\frac{1}{Q_c^2} - 1 \right) + \left(\frac{1}{Q_c^2} - 1 \right)^2 \right] \end{aligned}$$

This actually gives three values, namely

$$\mu_1 = 0 \text{ (max.)}, \quad \mu_2 = \sqrt{\frac{1}{3} \left(1 - \frac{1}{Q_c^2} \right)} \text{ (min.)},$$

$$\mu_3 = \sqrt{1 - \frac{1}{Q_c^2}} \text{ (max.)}$$

The last value gives the “resonant frequency” $\omega_{\text{res}} = \mu_3 \Omega_c$, because it corresponds to the maximum amplitude response of the network. Clearly, for undamped response $\omega_{\text{res}} = \Omega_c$; as the terminal dissipation increases, the resonant frequency decreases, until “resonance” actually disappears at $Q_c = 1$, or for

$$R = \sqrt{2} \sqrt{\frac{L}{C}}$$

We also notice that, as the dissipation increases, the response becomes flatter over a frequency band from zero to somewhat beyond the resonant response, a fully expected result.

The most characteristic quantities for the transient response are the natural modes given as y_2 and $y_{3,4}$ in (44). In particular, we are interested in the natural frequency associated with $y_{3,4}$ which we obtain as a function of the resistance from the roots of the first factor in (42). We use again as parameters $\Omega_c = (2/LC)^{1/2}$ and $Q_c = \Omega_c L/R$, so that from (42)

$$\left(\frac{p}{\Omega_c} \right)^2 + \frac{1}{Q_c} \left(\frac{p}{\Omega_c} \right) + 1 = 0$$

with the roots

$$p_{3,4} = \Omega_c \left(-\frac{1}{2Q_c} \pm j \sqrt{1 - \frac{1}{4Q_c^2}} \right)$$

The natural frequency is thus

$$\Omega' = \Omega_c \sqrt{1 - \frac{1}{4Q_c^2}}$$

It decreases with increasing dissipation, but at a very much smaller rate than the resonant frequency ω_{res} in the foregoing. To demonstrate this more vividly, we have plotted in Fig. 2.11 both quantities against $1/Q_c$. We see that even for considerable dissipation, down to $Q_c = 4$, the two characteristic frequencies differ but little, i.e., we can approximate one by the other. In addition, for $Q_c > 2$ we can choose the

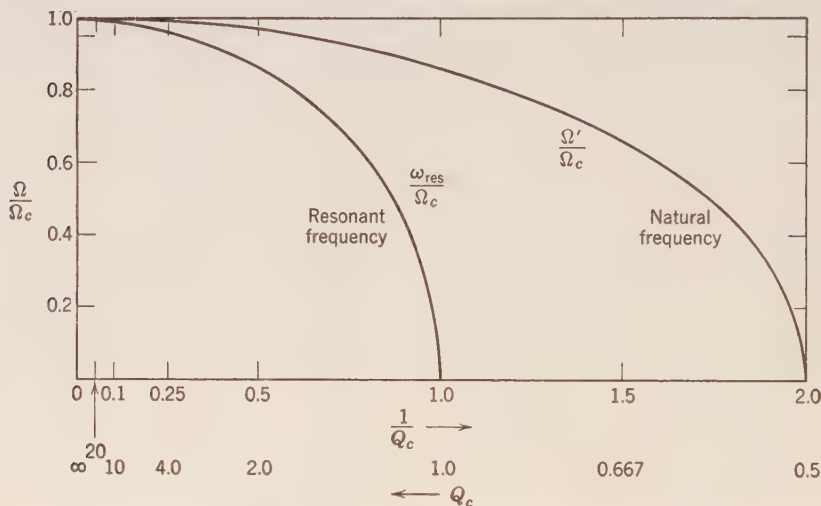


Fig. 2.11. Natural frequency Ω' and resonant frequency ω_{res} as functions of dissipation (network of Fig. 2.9).

cutoff or undamped free frequency of the network Ω_c as an excellent approximation to the actual natural frequency. This is a most welcome result as we shall demonstrate presently.

We have considered so far a single symmetrical fourpole. If we cascade two like fourpoles, we can determine the resulting general network parameters $\mathfrak{A}_{(2)}$, $\mathfrak{B}_{(2)}$, $\mathfrak{C}_{(2)}$, $\mathfrak{D}_{(2)}$ and conceivably use these in the basic forms. By matrix multiplication we have

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{A} \end{bmatrix}_{(2)} = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{A} \end{bmatrix} \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{A} \end{bmatrix} = \begin{bmatrix} 2\mathfrak{A}^2 - 1 & 2\mathfrak{A}\mathfrak{B} \\ 2\mathfrak{A}\mathfrak{C} & 2\mathfrak{A}^2 - 1 \end{bmatrix} \quad (47)$$

where the subscript (2) will be applied to all quantities pertaining to the cascade of two fourpoles. Now we can use the parameters from (47) directly in (39) and thus obtain for the Laplace transform of the

over-all output current

$$\begin{aligned} I_{2(2)}(p) &= \frac{2\alpha \mathfrak{C} V_0}{(2\alpha \mathfrak{C} Z_t + 2\alpha^2 - 1 + 1)(2\alpha \mathfrak{C} Z_t + 2\alpha^2 - 1 - 1)} \\ &= \frac{\mathfrak{C} V_0}{2(\mathfrak{C} Z_t + \alpha)(\alpha^2 + \alpha \mathfrak{C} Z_t - 1)} \end{aligned} \quad (48)$$

which is still quite simple in appearance. If we apply it, just for test, to the cascade of two T-sections, using the same terminology and assumptions as before, we find instead of (43)

$$\bar{I}_{2(2)}(p) = \frac{\Omega_c C V_m}{4} \frac{1}{(p - j\omega)(2y^3 + 2y^2 + 2y + 1)(2y^2 + 2y + 1)} \quad (49)$$

with the root values (in the order of factors in the denominator)

$$\begin{aligned} p_1 &= j\omega; & y_2 &= -0.65, & y_{3,4} &= -0.175 \pm j\sqrt{0.74}; \\ & & y_{5,6} &= -\frac{1}{2} \pm \frac{j}{2} \end{aligned} \quad (50)$$

There are now two natural frequencies, one very close to the value in (44), the other considerably lower so that even for small dissipation we shall expect a more uniform response over the low frequency range.

If we carry the process of cascading further, we find successively

$$\begin{bmatrix} \alpha & \mathfrak{B} \\ \mathfrak{C} & \alpha \end{bmatrix}_{(3)} = \begin{bmatrix} \alpha(4\alpha^2 - 3) & \mathfrak{B}(4\alpha^2 - 1) \\ \mathfrak{C}(4\alpha^2 - 1) & \alpha(4\alpha^2 - 3) \end{bmatrix} \quad (51)$$

$$\begin{bmatrix} \alpha & \mathfrak{B} \\ \mathfrak{C} & \alpha \end{bmatrix}_{(4)} = \begin{bmatrix} 8\alpha^4 - 8\alpha^2 + 1 & 4\alpha\mathfrak{B}(2\alpha^2 - 1) \\ 4\alpha\mathfrak{C}(2\alpha^2 - 1) & 8\alpha^4 - 8\alpha^2 + 1 \end{bmatrix} \quad (52)$$

A very simple method of obtaining the free frequencies of the simple fourpole or a short cascade of simple fourpoles can be deduced from the general fourpole relation (11). It we set $V_2 = 0$ in the short-circuit case, we have left only

$$V_1(p) = \mathfrak{B} I_2(p), \quad I_2(p) = \frac{V_1(p)}{\mathfrak{B}} \quad (53)$$

which is identical with (36) if we assume $Z_{t1} = Z_{t2} = 0$, i.e., short-circuit condition at both input and output terminal pairs. The roots of \mathfrak{B} must therefore give the natural frequencies of the system, short-circuited upon itself, and, as we have seen, these will be good approximations for resistive terminations. Let us take again the example of the low-pass T-section of Fig. 2.9 for which we have computed $\mathfrak{B}_{(1)}$ in (41). We get

$$\mathfrak{B}_{(1)} = pL(2 + p^2LC) = 0, \qquad \Omega_{(1)} = \frac{\sqrt{2}}{\sqrt{LC}} = \Omega_c$$

which is, of course, identical with Ω' for $Q_c = \infty$ in Fig. 2.11. For two T-sections in cascade we have from (47)

$$\mathfrak{B}_{(2)} = 2\mathfrak{A}_{(1)}\mathfrak{B}_{(1)} = 0$$

where $\mathfrak{B}_{(1)}$ is as above, and $\mathfrak{A}_{(1)} = 1 + p^2LC$ from (41). The two natural frequencies are then

$$\Omega_{(2)'} = \Omega_c, \qquad \Omega_{(2)''} = \Omega_c/\sqrt{2}$$

We take $\mathfrak{B}_{(3)}$ from (51) and $\mathfrak{B}_{(4)}$ from (52) and notice the great advantage of the product representation of $\mathfrak{B}_{(n)}$ so that there is no great problem in obtaining the undamped natural frequencies. Tabulating the results so as to exhibit the pattern, we have for the ratios Ω/Ω_c :

1 section	1.0				
2 sections	1.0		0.707		
3 sections	1.0		0.792	0.613	
4 sections	1.0	0.924	0.707		0.383

This demonstrates clearly how the addition of sections fills in natural frequencies between the top value Ω_c and zero and thus must approach the ideal low-pass filter with the cutoff frequency Ω_c . The interrelation between resonant frequencies of steady-state and natural frequencies of the transient state has been stressed by Guillemin,^{A8} Vol. II, pp. 285–287, as the physical aspect of the filter action.

A rather extensive treatment of specific simple fourpoles and cascades of two or more stages has been given* with many graphs of the actual transient responses to unit step voltage or current and to sinusoidal voltage. Most of the filter sections considered are actually coupling networks in television amplifiers where the transient problem has become particularly important.

2.4 Transformers

The rigorous treatment of transients in transformers is far beyond the scope of this book.† However, because the transformer is an

* M. E. Kallmann, R. E. Spencer, and C. P. Singer, "Transient Response," *Proc. I.R.E.*, **33**, 169–195 (1945).

† Rather extensive treatments, especially of power transformers where the problem is a serious one, can be found in R. Rüdenberg, *Transient Performance of Electric Power Systems*, McGraw-Hill, New York, 1950, chapters 8 and 9, with many references to the literature.

important element in all transmission systems, it might be well to consider some of its significant characteristics. The earlier and more extensive literature has been contributed in the power field, so that we shall first use the more conventional approach and then link the treatment to the generalized fourpole notation.

As in any system involving magnetic linkage, we must recognize the difference between "useful linkage" serving the transmission of power or intelligence and the "leakage," i.e., magnetic flux linked with only one part of the circuit and not participating in the transmission directly. We shall assume that all magnetic fluxes are proportional to the producing currents, which is justified even in iron-core transformers at low saturations. The primary winding of N_1 turns, as

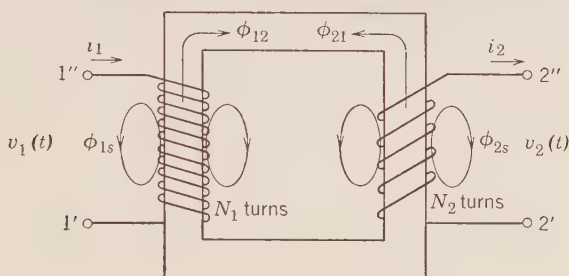


Fig. 2.12. Schematic of a transformer, $N_1/N_2 = 1/a$.

indicated in the schematic Fig. 2.12, will produce a total magnetic flux

$$\Phi_1 = \Phi_{12} + \Phi_{1S} = \varphi_{12}N_1i_1 + S_1i_1 \quad (54)$$

where Φ_{12} is that part linking the secondary winding and closely proportional to the number of turns so that φ_{12} can be considered as a geometric factor; and where $\Phi_{1S} = S_1i_1$ is its leakage part, certainly proportional to the current but not necessarily to the number of turns. Similarly we have for the secondary winding of N_2 turns

$$\Phi_2 = \Phi_{21} + \Phi_{2S} = \varphi_{21}N_2i_2 + S_2i_2 \quad (55)$$

If we now write the Kirchhoff mesh equation for the primary winding in terms of flux linkages, we have

$$R_1i_1 + N_1 \frac{d\Phi_{12}}{dt} + N_1 \frac{d\Phi_{21}}{dt} + \frac{d\Psi_{1S}}{dt} = v_1(t)$$

where we use Ψ_{1S} to designate the leakage flux linkages since we cannot generally assume that Φ_{1S} effectively links all N_1 turns. Similarly

$$R_2 i_2 + N_2 \frac{d\Phi_{21}}{dt} + N_2 \frac{d\Phi_{12}}{dt} + \frac{d\Psi_{2S}}{dt} = v_2(t)$$

Referring back to (54) and (55), we may now introduce the conventional inductances (assuming linear relationships as stated above),

$$\begin{aligned} L_1 &= \frac{1}{i_1} (N_1 \Phi_{12} + \Psi_{1S}) = \frac{N_1}{N_2} M + S_1 = L_{12} + S_1 \\ L_2 &= \frac{1}{i_2} (N_2 \Phi_{21} + \Psi_{2S}) = \frac{N_2}{N_1} M + S_2 = L_{21} + S_2 \end{aligned} \quad (56)$$

$$M = \varphi_{12} N_1 N_2 = \varphi_{21} N_2 N_1$$

From the first two definitions we obtain for the turns ratio, which we will designate as $1/a$,

$$\frac{N_1}{N_2} = \frac{1}{a} = \sqrt{\frac{L_{12}}{L_{21}}} = \sqrt{\frac{L_1 - S_1}{L_2 - S_2}}$$

Only if we have no leakage, so that $S_1 = S_2 = 0$, do we have

$$L_1 L_2 = M^2, \quad a = \sqrt{\frac{L_2}{L_1}} \quad \text{for } S_1 = S_2 = 0 \quad (57)$$

With leakage present, we define

$$M = k \sqrt{L_1 L_2}, \quad \sigma = 1 - k^2 \quad 0 \leq k \leq 1 \quad (58)$$

where k is the coupling coefficient and σ the leakage coefficient. Because of the nature of the leakage fluxes it is not possible to define primary and secondary leakage or coupling coefficients individually in any unique manner.

With the inductance definitions (56) we can rewrite the transformer differential equations in the usual form

$$\begin{aligned} R_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} &= v_1(t) \\ M \frac{di_1}{dt} + R_2 i_2 + L_2 \frac{di_2}{dt} &= v_2(t) \end{aligned} \quad (59)$$

Currents and voltages are the quantities actually measured in the primary and secondary, respectively. Frequently we find the statement that "all quantities have been referred to" either the primary or secondary side; this is useful in graphical analyses or in equivalent circuit diagrams and essentially means that the turns ratio has been

eliminated and the transformer can be treated as a 1:1 magnetic coupling. We can readily refer (59) to the primary by noting that $(N_1/N_2)M = L_{12}$ is measured on the primary side; to compensate, we need to measure current i_2 in the first line as $[(N_2/N_1)i_2] = i_2'$. To read the second line in quantities referred to the primary, we must again use $[(N_1/N_2)M] = M' = L_{12}$ but the current i_1 cannot be altered, i.e., we must multiply the whole equation by N_1/N_2 . We thus obtain

$$\begin{aligned} R_1 i_1 + L_1 \frac{di_1}{dt} + L_{12} \frac{di_2'}{dt} &= v_1(t) \\ L_{12} \frac{di_1}{dt} + R_2' i_2' + L_2' \frac{di_2'}{dt} &= v_2'(t) \end{aligned} \quad (60)$$

where

$$\begin{aligned} i_2' &= \left(\frac{N_2}{N_1}\right) i_2, & R_2' &= \left(\frac{N_1}{N_2}\right)^2 R_2, & L_2' &= \left(\frac{N_1}{N_2}\right)^2 L_2, \\ v_2'(t) &= \frac{N_1}{N_2} v_2(t) \end{aligned} \quad (61)$$

The corresponding adjustments can be made to reflect all quantities into the secondary.

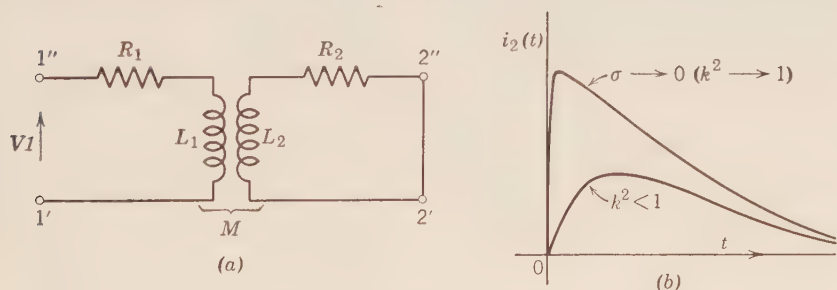


Fig. 2.13. Short-circuited transformer; (a) diagram, (b) response to step voltage.

To determine the characteristic transient behaviour of the transformer, let us apply a step voltage VI to the primary terminal pair with the secondary short-circuited, i.e., $V_2 = 0$ as in Fig. 2.13a. Taking the Laplace transform of the system (59), with zero initial energies, we have

$$\begin{aligned} (R_1 + pL_1)I_1 + pMI_2 &= \frac{V}{p} \\ pMI_1 + (R_2 + pL_2)I_2 &= 0 \end{aligned} \quad (62)$$

from which the secondary current transform follows

$$I_2(p) = \frac{-MV}{(L_1L_2 - M^2)p^2 + (R_1L_2 + R_2L_1)p + R_1R_2}$$

To simplify, we divide by L_1L_2 , use (58), and introduce

$$\alpha_1 = \frac{R_1}{L_1} = \frac{1}{T_1}, \quad \alpha_2 = \frac{R_2}{L_2} = \frac{1}{T_2} \quad (63)$$

as damping coefficients or reciprocal time constants, so that

$$I_2(p) = -\frac{k^2}{M} \frac{V}{\sigma p^2 + (\alpha_1 + \alpha_2)p + \alpha_1\alpha_2} \quad (64)$$

The roots of the denominator are

$$p_{1,2} = [-(\alpha_1 + \alpha_2) \pm \sqrt{(\alpha_1 - \alpha_2)^2 + 4k^2\alpha_1\alpha_2}] \frac{1}{2\sigma} \quad (65)$$

which can never be complex, bearing out the physical reasoning that oscillations are not possible in linear single energy networks. We actually see that, as $k \rightarrow 0$, $\sigma \rightarrow 1$ and the two root values become simply those of the individual, infinitely separate circuits. The time solution, i.e., the inverse Laplace transform, can be found best by the sum of the residues, namely

$$i_2(t) = \mathcal{L}^{-1}I_2(p) = -\frac{k^2}{M} \frac{V}{\sqrt{(\alpha_1 - \alpha_2)^2 + 4k^2\alpha_1\alpha_2}} [e^{p_1t} - e^{p_2t}] \quad (66)$$

For very tight coupling $k \rightarrow 1$ and $\sigma \rightarrow 0$, as in efficient power transformers and hopefully in the "ideal" transformer of network theory. We can write the terms under the square root in (65) also

$$(\alpha_1 + \alpha_2)^2 - 4\sigma\alpha_1\alpha_2$$

so that the root values from (65) become, since σ is small

$$p_{1,2} = -\frac{\alpha_1 + \alpha_2}{2\sigma} \left[1 \mp \left(1 - \frac{2\sigma\alpha_1\alpha_2}{(\alpha_1 + \alpha_2)^2} \right) \right] \approx \begin{cases} -\frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2} \\ -\frac{\alpha_1 + \alpha_2}{\sigma} \end{cases} \quad (67)$$

Letting $k^2 = 1$ in the amplitude factor, (66) becomes

$$i_2(t)|_{k^2 \rightarrow 1} = -\lim_{\sigma \rightarrow 0} \frac{V}{M(\alpha_1 + \alpha_2)} [e^{-\frac{t}{T}} - e^{-\frac{\alpha_1 + \alpha_2}{\sigma}t}] \quad (68)$$

The current is now a simple exponential with a time constant $T = (\alpha_1 + \alpha_2)/(\alpha_1\alpha_2) = T_1 + T_2$ which is the sum of the two separate time constants, but superimposed is a negative spike of unit height which just assures the current to start from zero with very high slope as indicated in Fig. 2.13*b*. If we had let $\sigma = 0$ in (64), we would have readily found (68) but containing only the first exponential.

A simplified equivalent circuit of the transformer is shown in Fig. 2.14 where all quantities are reflected into the primary, and where for further simplification we have assumed the reflected parameters to be entirely symmetrical. Here we shall apply a sinusoidal voltage $V e^{j\omega t}$ at the input terminal pair with the output pair again short-circuited.

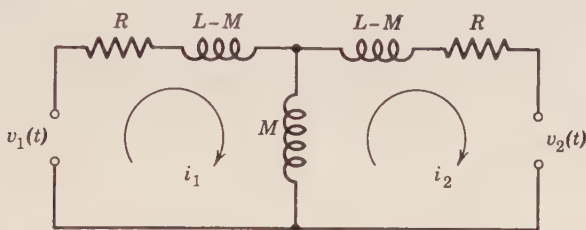


Fig. 2.14. Simplified equivalent circuit of symmetrical transformer.

In Laplace transform notation, combined with the complex notation for the applied voltage, we have for the initially de-energized state

$$\begin{aligned} (R + pL)\bar{I}_1 - pM\bar{I}_2 &= \frac{V}{p - j\omega} \\ -pM\bar{I}_1 + (R + pL)\bar{I}_2 &= 0 \end{aligned} \quad (69)$$

The negative sign for the mutual term arises naturally from the choice of the loop currents. Though it is different from the one used in (62) which was based on the concept of the flux linkages, either convention is equally useful and “correct” and neither need arouse long, justifying discussions. The solution for the primary current transform gives

$$\bar{I}_1 = \frac{V}{(p - j\omega)L} \frac{p + \alpha}{\sigma p^2 + 2\alpha p + \alpha^2} \quad (70)$$

using the same notation as (63), except that now $\alpha_1 = \alpha_2$. The root values are simply from (65)

$$p_{1,2} = -\frac{\alpha}{\sigma} \pm \frac{k\alpha}{\sigma} = -\frac{\alpha}{1+k}, -\frac{\alpha}{1-k}$$

Further, since $\alpha = R/L$, $k = M/L$

$$p_1 = -\frac{R}{L+M} = -\frac{1}{T_L}, \quad p_2 = -\frac{R}{L-M} = -\frac{1}{T_S} \quad (71)$$

T_L will be a long time constant, related to a large inductance ($L+M$), whereas T_S will be short and depend on the presence of leakage since $L-M=S$ from (56) for $N_1=N_2$.

The inverse transform of (70) is in complex notation,

$$\bar{i}_1(t) = \frac{V}{L} \left(\frac{j\omega + \alpha}{-\sigma\omega^2 + 2j\alpha\omega + \alpha^2} e^{j\omega t} + \frac{1}{2(p_1 - j\omega)(1+k)} e^{-\frac{t}{T_L}} + \frac{1}{2(p_2 - j\omega)(1-k)} e^{-\frac{t}{T_S}} \right)$$

For any specific phase angle of the applied voltage one can readily carry the rationalization further. It will suffice here to compute the amplitude of the steady-state component, which is given by the first term above,

$$|\bar{I}_{1ss}| = |V| \frac{\sqrt{R^2 + (\omega L)^2}}{\{[R^2 - \sigma(\omega L)^2]^2 + (2R\omega L)^2\}^{1/2}}$$

If the resistance is small, as it invariably is in power transformers, we can disregard it as compared with the inductive reactances and we can thus approximate

$$|\bar{I}_{1ss}| \approx \frac{|V|}{\sigma\omega L}$$

With V the rated voltage value and ωL the approximate normal input impedance of the primary circuit, the rated current can be taken as $V/\omega L$. The sustained terminal short-circuit current might then be $1/\sigma$ times the rated current; for $\sigma = 8\%$ this gives 12.5 times normal current. Instantaneously, we must add the transient terms which might almost double this already large value. The forces upon windings in power transformers and electric machines under sudden short circuit are therefore enormous. For extensive discussions see Rüdenberg, *op. cit.*, chapters 9 and 12 through 15, with many references to the literature. In *audio transformers*, where it might be more proper to assume R of the same order as ωL , we find the sustained short-circuit current only slightly higher than the normal rated current.

A representation of the transformer in terms of the general fourpole parameters proceeds best from the system of transform equations patterned after (69)

$$\begin{aligned}(R_1 + pL_1)I_1 - pMI_2 &= V_1(p) \\ -pMI_1 + (R_2 + pL_2)I_2 &= -V_2(p)\end{aligned}$$

which are also in the same form as (9) for the basic T-structure of Fig. 2.3b. Solving for the input current and voltage

$$\begin{aligned}V_1 &= \frac{R_1 + pL_1}{pM} V_2 + \frac{(R_1 + pL_1)(R_2 + pL_2) - p^2M^2}{pM} I_2 \\ I_1 &= \frac{1}{pM} V_2 + \frac{R_2 + pL_2}{pM} I_2\end{aligned}\quad (72)$$

We can read off the parameters \mathfrak{A} , \mathfrak{B} , \mathfrak{C} , \mathfrak{D} directly; because of the choice of negative sign for the mutual coupling terms, all parameters have positive signs which is desirable.

In many network applications *lossless transformers* are inserted, for which we have from (72) with $R_1 = R_2 = 0$

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{\text{Tr}} = \begin{bmatrix} \frac{L_1}{M} & pM \left(\frac{1}{k^2} - 1 \right) \\ \frac{1}{pM} & \frac{L_2}{M} \end{bmatrix} \quad (73)$$

If, in addition, the transformer has no leakage, it is called an *ideal transformer*. Definitions (56) show readily that the terms along the main diagonal in (73) reduce to the turns ratio a and its inverse, and with $k = 1$ we see $\mathfrak{B} = 0$; the troublesome term is \mathfrak{C} . Going back to the transformer equations in terms of fluxes preceding (56), we see that for $R_1 = R_2 = 0$ and $\Psi_{1s} = \Psi_{2s} = 0$ we have simply

$$\frac{v_1(t)}{N_1} = \frac{v_2(t)}{N_2} = \frac{d}{dt} (\Phi_{12} + \Phi_{21}) \quad (74)$$

or

$$V_1(p) = \frac{N_1}{N_2} V_2(p) = \frac{1}{a} V_2(p)$$

in Laplace transforms. But (74) demands from energy considerations that

$$N_1 \dot{i}_1(t) = N_2 \dot{i}_2(t)$$

so that

$$I_1(p) = \frac{N_2}{N_1} I_2(p) = a I_2(p)$$

Accepting these two relations as the defining equations for the ideal transformer equivalent to (72) for the real transformer, we then get

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{\text{Tr, ideal}} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & a \end{bmatrix} \quad (75)$$

If a fourpole is supplied through an ideal transformer as in Fig. 2.15, then the over-all matrix of general parameters will be given by

$$\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & a \end{bmatrix} \times \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix} = \begin{bmatrix} \frac{\mathfrak{A}}{a} & \frac{\mathfrak{B}}{a} \\ a\mathfrak{C} & a\mathfrak{D} \end{bmatrix}$$

Further discussion of ideal transformers in networks is found in Guillemin,^{A8} Vol. II, pp. 151–156.

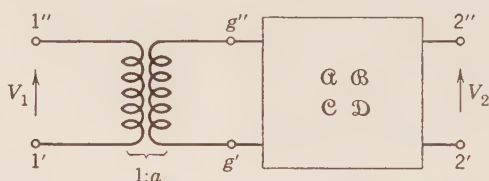


Fig. 2.15. Fourpole with ideal transformer input.

2.5 Magnetically Coupled Circuits

In general, the magnetic coupling of two coils as elements of a network can be supporting or counteracting the individual coil current. It has become customary to use a dot symbol with each coil in addition to the (arbitrary) choice of current direction to indicate the mutual inductive action as either positive or negative and always to ascribe to M , the mutual induction coefficient, positive values. Thus, in the simple coupled circuit of Fig. 2.16a, the fact that the dot in the primary coil is at the point where the current, as chosen, enters, whereas in the secondary coil the dot is placed at the point where i_2 , as chosen, leaves that coil, indicates that the mutual inductive action is negative. Had both dots been placed at the respective current entrance sides of the coils, then the mutual inductive action would be assumed positive. Obviously this convention can be used in ordinary transformer circuits also, though actually little is gained by it.

The simple coupled circuits of Fig. 2.16a, taken as a fourpole, present the same aspect as the transformer of Fig. 2.13 with the capacitances C_1 and C_2 replacing the corresponding resistances. We

can therefore write the general parameters directly from (72) as

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{CC} = \frac{1}{pM} \begin{bmatrix} pL_1 + \frac{1}{pC_1} & \left(pL_1 + \frac{1}{pC_1}\right)\left(pL_2 + \frac{1}{pC_2}\right) - p^2M^2 \\ 1 & pL_2 + \frac{1}{pC_2} \end{bmatrix} \quad (76)$$

where the factor $1/(pM)$ applies to all terms of the matrix. We could readily check this as the cascade arrangement of three fourpoles, the two outer ones being simple series impedances $1/(pC_1)$ and $1/(pC_2)$,

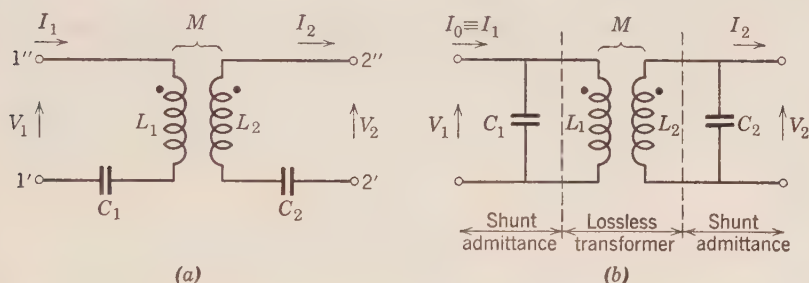


Fig. 2.16. Two coupled circuits without losses; (a) series circuits, (b) parallel circuits.

the middle one being the lossless transformer with matrix (73). The underlying mesh equations with the directions of currents and voltages as in Fig. 2.16a are in Laplace transform notation

$$\begin{aligned} \left(pL_1 + \frac{1}{pC_1}\right) I_1(p) - pMI_2(p) &= V_1(p) \\ -pMI_1(p) + \left(pL_2 + \frac{1}{pC_2}\right) I_2(p) &= -V_2(p) \end{aligned} \quad (77)$$

from which we can obtain the coefficient matrix (76) by solving for the pair V_1, I_1 .

To find the characteristic transient response* in the simplest form, we again apply a step voltage $v_1(t) = VI$ and short-circuit the output

* An excellent analogy is given by P. C. Magnusson, "Transients in Coupled Inductance-Capacitance Circuits Analyzed in Terms of a Rolling-Ball Analogue," *Trans. AIEE*, **69**, 1525-30 (1950). See also mechanical analogies in K. W. Wagner, "Natural Oscillations and Damping of Coupled Vibrators" [German], *Telegr.-u. Fernspr. Tech.*, **24**, 191-203 (1935).

terminals $v_2 = 0$. Either from (77), or from (76) if we refer back to the general solution (53), we find

$$I_2(p) = \frac{V_1(p)}{\mathfrak{B}} = \frac{pMV}{p \left[\left(pL_1 + \frac{1}{pC_1} \right) \left(pL_2 + \frac{1}{pC_2} \right) - p^2M^2 \right]}$$

If we introduce k^2 and σ from (58) as well as

$$\Omega_1 = \frac{1}{\sqrt{L_1C_1}}, \quad \Omega_2 = \frac{1}{\sqrt{L_2C_2}} \quad (78)$$

the expression simplifies to

$$I_2(p) = + \frac{k^2}{M} V \frac{p^2}{\sigma p^4 + (\Omega_1^2 + \Omega_2^2)p^2 + \Omega_1^2\Omega_2^2} \quad (79)$$

This compares with (64) for the transformer where the change in sign is due to the difference in sign for the mutual coupling term in the mesh equations. We can write the roots of the denominator in analogy to (65)

$$p_{1,2}^2 = [-(\Omega_1^2 + \Omega_2^2) \pm \sqrt{(\Omega_1^2 - \Omega_2^2)^2 + 4k^2\Omega_1^2\Omega_2^2}] \frac{1}{2\sigma} \quad (80)$$

Both p_1^2 and p_2^2 will always be negative real, so that the transient response consists of sustained oscillations of the frequencies

$$\Omega' = \sqrt{-p_1^2}, \quad \Omega'' = \sqrt{-p_2^2}$$

For the complete solution it is worth while to use the partial fraction rule and to write (79)

$$I_2(p) = \frac{k^2}{M} V \frac{1}{\sigma} \left(\frac{B_1}{p^2 - p_1^2} + \frac{B_2}{p^2 - p_2^2} \right)$$

with the B_1 and B_2 constants determined by conventional coefficient comparison with (79). We find

$$B_1 = \frac{p_1^2}{p_1^2 - p_2^2}, \quad B_2 = \frac{-p_2^2}{p_1^2 - p_2^2}$$

Identifying the quadratic fractions from Table 1.3 we obtain for the complete solution

$$i_2(t) = \frac{k^2}{M} \frac{V}{[(\Omega_1^2 - \Omega_2^2)^2 + 4k^2\Omega_1^2\Omega_2^2]^{1/2}} (\Omega'' \sin \Omega''t - \Omega' \sin \Omega't) \quad (81)$$

The parallel circuit arrangement of Fig. 2.16*b* is used more frequently in vacuum-tube circuits with a current source; the losses which could be taken into account by shunt conductances are disregarded. We can consider this network as the cascade connection of three elementary fourpoles, capacitive admittances like (22) on the outside and the lossless transformer (73) in the center. Performing the matrix multiplications

$$\begin{bmatrix} 1 & 0 \\ pC_1 & 1 \end{bmatrix} \times \begin{bmatrix} \frac{L_1}{M} & pM \left(\frac{1}{k^2} - 1 \right) \\ \frac{1}{pM} & \frac{L_2}{M} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ pC_2 & 1 \end{bmatrix}$$

and rearranging the terms, we arrive at the final forms

$$\begin{bmatrix} \frac{L_1}{M} (1 + \sigma p^2 L_2 C_2) & pM \left(\frac{1}{k^2} - 1 \right) \\ \frac{1}{pM} [1 + p^2 (L_1 C_1 + L_2 C_2) + p^4 \sigma L_1 L_2 C_1 C_2] & \frac{L_2}{M} (1 + \sigma p^2 L_1 C_1) \end{bmatrix} \quad (82)$$

Here we want to solve for the output voltage $v_2(t)$ in terms of the applied current $i_1(t) \equiv i_0(t)$. With the condition $I_2 = 0$ we take directly from the second of the general fourpole equations (11) or also from the definition (17)

$$V_2(p) = \frac{I_1(p)}{\mathfrak{C}} = \frac{I_0(p)}{\mathfrak{C}} \quad (83)$$

where $I_0(p)$ is the Laplace transform of the impressed current $i_0(t)$. Whereas with the voltage source and $V_2 = 0$ the general parameter $1/\mathfrak{B} = -Y_{21}$ was of paramount importance, here with the current source and $I_2 = 0$ the general parameter $1/\mathfrak{C} = Z_{21}$ is of basic importance.

Applying (83) with \mathfrak{C} from (82), and assuming a step current $I_0 I$ with Laplace transform I_0/p impressed, we have

$$V_2(p) = \frac{pMI_0}{p[\sigma p^4 L_1 L_2 C_1 C_2 + p^2 (L_1 C_1 + L_2 C_2) + 1]}$$

The introduction of Ω_1, Ω_2 from (78), taking $L_1 L_2 C_1 C_2$ outside the denominator and recalling $k^2 = M^2/L_1 L_2$, reduces this to

$$V_2(p) = \frac{k^2 I_0}{MC_1 C_2 \sigma p^4 + p^2 (\Omega_1^2 + \Omega_2^2) + \Omega_1^2 \Omega_2^2} \quad (84)$$

The denominator is identical with that in (79) so that the same root values (80) apply. We use again the partial fraction expansion and obtain as the final time solution

$$v_2(t) = \frac{k^2 I_0}{MC_1 C_2 [(\Omega_1^2 - \Omega_2^2)^2 + 4k^2 \Omega_1^2 \Omega_2^2]^{\frac{1}{2}}} \left[\frac{1}{\Omega'} \sin \Omega' t - \frac{1}{\Omega''} \sin \Omega'' t \right] \quad (85)$$

which is quite similar to the response of the series circuits.

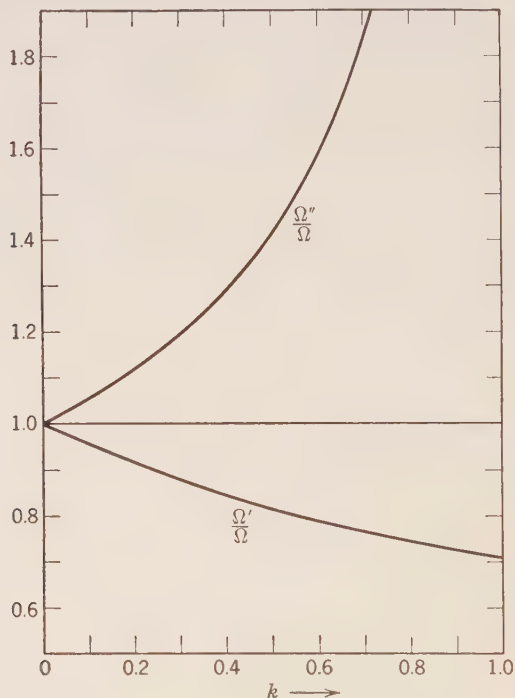


Fig. 2.17. Natural frequencies of the coupled circuits of Fig. 2.16 as functions of the coupling coefficient k .

Of particular importance is the condition for which $\Omega_1 = \Omega_2 = \Omega$. In this case, the natural frequencies are from (80)

$$\Omega' = \frac{\Omega}{\sqrt{1+k}}, \quad \Omega'' = \frac{\Omega}{\sqrt{1-k}} \quad (86)$$

Fig. 2.17 gives these natural frequencies as functions of the coupling coefficient k . It is obvious that even for very loose coupling the two natural frequencies will be different from the natural frequency of the

individual circuits. If we choose complete symmetry so that also

$$L_1 = L_2 = L, \quad C_1 = C_2 = C, \quad \Omega = \frac{1}{\sqrt{LC}}, \quad k = \frac{M}{L}$$

then (85) reduces with (86) to the much simpler form

$$v_2(t) = I_0 \sqrt{\frac{L}{C}} \frac{1}{2} (\sqrt{1+k} \sin \Omega t - \sqrt{1-k} \sin \Omega' t)$$

which is still valid for any value of k . For $k \ll 1$ we may use the binomial expansions

$$\frac{\Omega}{\sqrt{1 \pm k}} \approx \Omega \left(1 \mp \frac{k}{2} \right)$$

suppress the k in the amplitude radicals, and combine the difference of the two sine functions to give

$$v_2(t) = -I_0 \sqrt{\frac{L}{C}} \cos \Omega t \sin \left(\frac{k}{2} \Omega t \right) \quad (87)$$

This form exhibits the angular beat frequency $[(k/2)\Omega]$ so that the degree of coupling can readily be determined from the observed beat note.*

Damping effects can be taken into account in a first approximation by simply attaching the exponential factors $e^{-\delta' t}$ and $e^{-\delta'' t}$, respectively, to the time functions in (81) and (85) and by choosing $\delta' = R_1/(2L_1)$, $\delta'' = R_2/(2L_2)$ if R_1 and R_2 are the corresponding small resistances in primary and secondary circuits. For a better approximation we can use the method outlined in Vol. I, section 5.10, which alters the root values and, of course, leads to considerably more involved expressions.

The application of a sinusoidal current at the input terminals of Fig. 2.16*b* does not alter the transient modes of response; it affects only their amplitudes. Defining the impressed current

$$i_0(t) = \text{Im} (Ie^{j\omega t}) = I_m \sin (\omega t + \psi) \quad (88)$$

with the phasor $I_m e^{j\psi} = I$, we have for its complex Laplace transform

$$\bar{I}_0(p) = \mathcal{L}(Ie^{j\omega t}) = \frac{I}{p - j\omega} \quad (89)$$

* J. van Slooten, "Experimental Testing of Electrical Networks by means of the Unit Function Response," *Philips Tech. Rev.*, **12**, 238 (Feb. 1951).

This has to be used now in (83), so that, going right to (84) and replacing I_0/p by (89), we get for the Laplace transform of the output voltage, again in complex notation,

$$\bar{V}_2(p) = \frac{k^2 I}{MC_1 C_2} \frac{1}{p - j\omega} \frac{p}{\sigma p^4 + p^2(\Omega_1^2 + \Omega_2^2) + \Omega_1^2 \Omega_2^2} \quad (90)$$

To obtain the inverse Laplace transform, we could use partial fraction expansion as before; however, we should find this *much* less convenient than for (84). Rather let us use here the convolution integral or Borel theorem from Table 1.5. We define the two factors as separated in (90) and have, respectively

$$F_1(p) = \frac{1}{p - j\omega}, \quad f_1(t) = e^{j\omega t}$$

$$F_2(p) = \frac{I}{I_0} p[V_2(p) \text{ from (84)}], \quad f_2(t) = \frac{I}{I_0} \frac{d}{dt} [v_2(t) \text{ from (85)}] \quad (91)$$

The first inverse Laplace transform is easily taken from Table 1.3; the second one we identify from Table 1.4 as the derivative since $v_2(0) = 0$. The complete inverse transform is then, leaving out the constant factors

$$\int_{\tau=0^+}^t e^{j\omega(t-\tau)} (\cos \Omega' \tau - \cos \Omega'' \tau) d\tau$$

which is readily evaluated (preferably using the equivalent exponential forms for the trigonometric functions). If we disregard any initial phase angle in (88), so that $\psi = 0$ and $I = I_m$, the real amplitude, we can take the imaginary part of the above integral and obtain for the complete solution

$$v_2(t) = \frac{k^2 I_m}{MC_1 C_2} \frac{1}{[(\Omega_1^2 - \Omega_2^2)^2 + 4k^2 \Omega_1^2 \Omega_2^2]^{\frac{1}{2}}} \left(\frac{\Omega''^2 - \Omega'^2}{(\omega^2 - \Omega'^2)(\omega^2 - \Omega''^2)} \cos \omega t + \frac{\omega}{\omega^2 - \Omega'^2} \cos \Omega' t - \frac{\omega}{\omega^2 - \Omega''^2} \cos \Omega'' t \right) \quad (92)$$

This clearly shows that the resonant amplitudes of the steady-state response (of infinite magnitude because of the disregard of losses) occur at the natural frequencies of the system and that the oscillatory transient terms also assume infinite amplitudes. Either network in

Fig. 2.16 behaves as a *band-pass filter*, the pass band being essentially defined by $(\Omega'' - \Omega')$, the exact definition depending on the arbitrary amplitude values selected for the delineation of "pass" and "rejection."

The case of "*tuned circuits*," is of particular interest again with $\Omega_1 = \Omega_2 = \Omega$ for which (86) defines the natural frequencies. We can reduce (92) to the simpler form

$$v_2(t) = \frac{k^2 I_m}{MC_1 C_2} \left[\frac{\omega}{\sigma \omega^4 - 2\omega^2 \Omega^2 + \Omega^4} \cos \omega t + \frac{\omega}{2k\Omega^2} \left(\frac{1+k}{(1+k)\omega^2 - \Omega^2} \cos \Omega' t - \frac{1-k}{(1-k)\omega^2 - \Omega^2} \cos \Omega'' t \right) \right]$$

The steady-state amplitude can easily be checked by introducing $p = j\omega$ and deleting the factor $1/(p - j\omega)$ in (90), i.e., by taking the residue at the pole $p = j\omega$. If damping is introduced, the infinite amplitudes become finite and one can explore the steady-state response as a function of the coupling coefficient k . Obviously, instead of varying k to affect the shape of the response curve, one could also stagger-tune the individual circuits by controlled amounts. Rather thorough discussions of the steady characteristics with damping can be found in many of the references on network theory and design. A very thorough classical discussion of the transient behaviour of coupled circuits is given in Pierce.*

Should we want to use two or more of these band-pass filter sections in cascade, we could apply the same method as in section 2.3 and in particular use the combination matrices (47), (51), and (52). For the network in Fig. 2.16*b* we had to use the general parameter \mathfrak{C} in (83) to compute the output voltage of the single section. Therefore, we need only to use successively

$$\mathfrak{C}_{(2)} = 2\mathfrak{C}_{(1)}\mathfrak{Q}_{(1)}$$

$$\mathfrak{C}_{(3)} = \mathfrak{C}_{(1)}(4\mathfrak{Q}_{(1)}^2 - 1) = \mathfrak{C}_{(1)}(2\mathfrak{Q}_{(1)} - 1)(2\mathfrak{Q}_{(1)} + 1)$$

$$\mathfrak{C}_{(4)} = 4\mathfrak{C}_{(1)}\mathfrak{Q}_{(1)}(2\mathfrak{Q}_{(1)}^2 - 1) = 4\mathfrak{C}_{(1)}\mathfrak{Q}_{(1)}(\mathfrak{Q}_{(1)}\sqrt{2} - 1)(\mathfrak{Q}_{(1)}\sqrt{2} + 1)$$

to evaluate the additional natural frequencies provided conveniently by the individual factors in the product sequence. Since from (82) we have

$$\mathfrak{Q}_{(1)} = \frac{L_1}{M} (1 + \sigma p^2 L_2 C_2)$$

* G. W. Pierce, *Electric Oscillations and Electric Waves*, McGraw-Hill, New York, 1920, Chapters VII through XI.

we can construct a table of frequencies very similar to the one given at the end of section 2.3.

2.6 Envelope Response to Amplitude-Modulated Signals

So far we have only considered simple impressed voltages and currents with the main purpose of ascertaining the transient response as distinct and separate from the steady-state response. Experimental observation usually gives a combination of both and it is frequently desirable to determine a criterion for the build-up time of the steady state without explicitly evaluating the natural response of the circuit itself. This will be particularly advantageous if we are dealing with amplitude-modulated carrier transmission and are primarily interested in the recovery of the modulation function which contains the signal.

Let us denote the carrier as $\sin \omega_0 t$, of fixed frequency, and consider the amplitude modulation defined by

$$m(t) \sin \omega_0 t = \text{Im} [m(t)e^{j\omega_0 t}] \quad (93)$$

where $m(t)$ is a *real* time function so that we can employ complex notation as indicated. We could, of course, also express $\sin \omega_0 t$ in terms of complex exponentials, but then we would have to carry along two terms.

To arrive at a general treatment, let us apply (93) as a source function to the input terminals of a fourpole and use either (53) or (83) depending upon convenience, or in general notation

$$\mathcal{R}(p) = \frac{\mathcal{S}(p)}{N(p)} \quad (94)$$

where $\mathcal{R}(p)$ is the Laplace transform of the unknown "response," $\mathcal{S}(p)$ that of the source function, $N(p)$ the parametric network function.

The Laplace transform of the amplitude-modulated source function in the complex form can be evaluated as follows

$$\mathcal{L}m(t)e^{j\omega_0 t} = \int_{0+}^{\infty} m(t)e^{j\omega_0 t}e^{-pt} dt = \int_{0+}^{\infty} m(t)e^{-p't} dt = M(p') \quad (95)$$

where $p' = p - j\omega_0$ and $M(p)$ is the conventional designation of the Laplace transform of the real time function $m(t)$, assuming, of course, that it exists. Actually, (95) is a general relationship, and can be stated as

$$\mathcal{L}f(t)e^{\gamma t} = F(p - \gamma) \quad (96)$$

It has been entered into Table 1.4, line 5; γ can have any value, real or complex, as long as $\text{Re}(p - \gamma) > 0$ so as to provide proper convergence in the Laplace transformation.

We thus have in (94)

$$\bar{\mathfrak{R}}(p) = \frac{M(p - j\omega_0)}{N(p)} \quad (97)$$

where the bar over the left-hand term merely reminds us that we need to take the imaginary part only. In order to recover the carrier wave in its original form and thus have the response as time function again expressed as the product of a modulation function and $e^{j\omega_0 t}$, we observe that the inverse Laplace operation on (96) must be valid, i.e., if we have a transform of type $F(p - \gamma)$, its inverse must be the time function on the left-hand side. This means then, applied to (97),

$$\mathfrak{L}^{-1} \frac{M(p - j\omega_0)}{N[(p + j\omega_0) - j\omega_0]} = e^{j\omega_0 t} \mathfrak{L}^{-1} \frac{M(p)}{N(p + j\omega_0)} = \bar{r}(t) \quad (98)$$

We obtain the response function in complex form by finding the inverse Laplace transform of the quotient of the direct Laplace transform of the *real modulation function* and the adjusted network function! Obviously, because of the adjustment, this inverse transform as time function will still be *complex*; let us define it

$$\bar{r}(t) = \mu(t) \exp [j\eta(t)] e^{j\omega_0 t} \quad (99)$$

The actual time response becomes, as the imaginary part of (99)

$$r(t) = \mu(t) \sin [\omega_0 t + \eta(t)] \quad (100)$$

The variable phase function $\eta(t)$ will have the effect of a phase modulation, or variable frequency, with the instantaneous frequency defined by

$$\omega_i = \frac{d}{dt} [\omega_0 t + \eta(t)] = \omega_0 + \frac{d}{dt} \eta(t)$$

This method will be particularly useful when we deal with idealized network functions, as will be discussed in section 4.

For lumped-parameter networks, the building up of sinusoidal currents can be described by the envelope method if we compare the total response with the steady-state sinusoidal oscillation as follows. If we have given a general impedance or admittance function $W(p)$, we might write (94)

$$\mathfrak{R}(p) = W(p) \mathfrak{S}(p) \quad (101)$$

with the source function $\mathfrak{S}(p)$ defined in real notation

$$\mathfrak{S}(p) = \mathfrak{L} \sin \omega_0 t = \frac{\omega_0}{p^2 + \omega_0^2}$$

The steady-state response is then

$$r_{ss}(t) = \text{Im} [W(j\omega_0)e^{j\omega_0 t}]$$

Suppose we write the complete response, including the transient and steady states, i.e., the building-up process

$$r(t) = \text{Im} \{W(j\omega_0)e^{j\omega_0 t}[1 + \rho(t) + j\sigma(t)]\} \quad (102)$$

then

$$\text{for } t \rightarrow \infty: \quad \rho(t) \rightarrow 0, \quad \sigma(t) \rightarrow 0$$

Or, if we rationalize and write this in a form similar to (100), except for keeping separate the steady-state immittance $W(j\omega_0)$, we get

$$r(t) = |W(j\omega_0)| \{[1 + \rho(t)]^2 + \sigma^2(t)\}^{1/2} \sin \left(\omega_0 t + \Phi(\omega_0) + \tan^{-1} \frac{\sigma(t)}{1 + \rho(t)} \right) \quad (103)$$

The factor $\mu(t)$ with the limit

$$\lim_{t \rightarrow \infty} \{[1 + \rho(t)]^2 + \sigma^2(t)\}^{1/2} \rightarrow 1$$

describes the *envelope* of the response, and the angle $\eta(t)$ with the limit

$$\lim_{t \rightarrow \infty} \tan^{-1} \frac{\sigma(t)}{1 + \rho(t)} \rightarrow 0$$

describes the phase modulation, or apparent variation of frequency.

To determine the envelope functions $\rho(t)$ and $\sigma(t)$, we define the complete response by means of Laplace transforms from (101)

$$r(t) = \mathcal{L}^{-1}W(p) \frac{\omega_0}{p^2 + \omega_0^2} \quad (104)$$

The steady-state part is given by the residues at the poles $p = \pm j\omega_0$ and must, of course, be identical with $r_{ss}(t)$, which also occurs in (102). The transient response is given by the singularities of the network function $W(p)$ and will be the sum of the pertinent residues for lumped-parameter networks. We can therefore take, by comparing (102) and (104), for the transient response

$$r_{tr}(t) = \text{Im} \{W(j\omega_0)e^{j\omega_0 t}[\rho(t) + j\sigma(t)]\} = \mathcal{L}^{-1}W(p) \frac{\omega_0}{p^2 + \omega_0^2} \quad (105)$$

(excluding $p = \pm j\omega_0$)

Because of the real notation in (104), the transient response will also appear in real form, but can always be redefined as

$$r_{tr}(t) = \text{Im} \bar{r}_{tr}(t)$$

if we just use complex exponentials for the sinusoidal time functions that appear. Thus, we can obtain explicitly from (105)

$$\rho(t) + j\sigma(t) = \frac{\bar{r}_{tr}(t)}{W(j\omega_0)e^{j\omega_0 t}} = \frac{\bar{r}_{tr}(t)}{\bar{r}_{ss}(t)} \quad (106)$$

As a simple illustration let us apply a step function modulated sinusoidal carrier current to the parallel tuned circuit of Fig. 2.18.

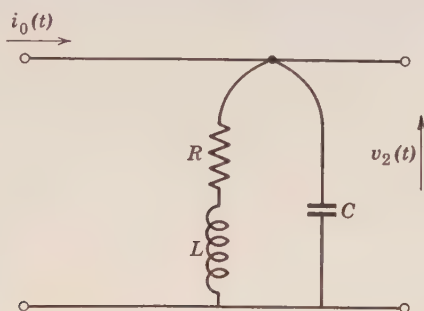


Fig. 2.18. Illustrative circuit for evaluation of envelope response.

The source function is

$$i(t) = I_0 \sin \omega_0 t = \text{Im} (I_0 e^{j\omega_0 t})$$

The output quantity is the voltage $v_2(t)$ across the node pair, so that we shall choose $W(p)$ as the impedance function in this case,

$$\Re(p) = V_2(p) = Z(p)I(p)$$

with

$$Z(p) = \frac{R + pL}{1 + pCR + p^2LC}, \quad I(p) = \frac{\omega_0 I_0}{p^2 + \omega_0^2}$$

Introducing the abbreviations

$$LC = \Omega_0^{-2}, \quad \frac{R}{L} = 2\alpha$$

and the dimensionless ratios

$$\frac{p}{\Omega_0} = x, \quad \frac{\alpha}{\Omega_0} = u, \quad \frac{\omega_0}{\Omega_0} = y$$

we have

$$V_2(p) = yLI_0 \frac{x + 2u}{(x^2 + y^2)(x^2 + 2ux + 1)}$$

We only take the residues at the poles of the network function,

$$x_{1,2} = -u \pm j \sqrt{1 - u^2}$$

It is obvious that $v_{2tr}(t)$ will be the sum of two conjugate complex time functions because the root values are conjugate complex. We can write it

$$v_{2tr}(t) = \omega_0 L I_0 \frac{e^{-\alpha t}}{\sqrt{1 - u^2}} \operatorname{Im} \frac{u + j \sqrt{1 - u^2}}{(-1 + 2u^2 + y^2) - 2ju \sqrt{1 - u^2}} e^{j\Omega t}$$

The steady-state solution for the output voltage is

$$v_{2ss}(t) = \operatorname{Im} (Z(j\omega_0) I_0 e^{j\omega_0 t})$$

so that (106) gives here

$$\begin{aligned} \rho(t) + j\sigma(t) &= \frac{\omega_0 L}{Z(j\omega_0)} \frac{e^{-\alpha t}}{\sqrt{1 - u^2}} \\ &\quad \frac{u + j \sqrt{1 - u^2}}{(-1 + 2u^2 + y^2) - 2ju \sqrt{1 - u^2}} e^{j(\Omega - \omega_0)t} \end{aligned} \quad (107)$$

Rather than carry this rather complicated expression further, let us remember that we have a tuned circuit so that we can assume $\omega_0 = \Omega \approx \Omega_0$, $y = 1$. If we define for the circuit

$$Q = \frac{\Omega_0 L}{R} = \frac{\Omega_0}{2\alpha} = \frac{1}{2u}, \quad u = \frac{1}{2Q}$$

we see that u might be quite small at reasonably high frequencies, so that we may assume $u^2 \ll 1$. This simplifies (107) very considerably, and we get

$$\rho(t) + j\sigma(t) \cong e^{-\alpha t} (-1 - j0)$$

The envelope of the building up will thus be approximately

$$\begin{aligned} (1 - e^{-\alpha t}) &\rightarrow 1 && \text{for } t \rightarrow \infty \\ &\rightarrow 0 && \text{for } t \rightarrow 0 \end{aligned}$$

If a frequency $\omega_0 \neq \Omega$ is applied, then (107) demonstrates that oscillations of the amplitude with a frequency $(\Omega - \omega_0)$ will occur, leading to a distinct overshoot whose magnitude can be computed from the envelope function. Fig. 2.19 shows the envelope response for the values $Q = 10$, $y = 0.97$. For the computation we observe that in (107) only the exponential functions contain the time explicitly. If

we therefore introduce $p = j\omega_0$ into $Z(p)$ and make use of the definitions of u and y , we can write

$$\frac{\omega_0 L}{Z(j\omega_0)} = \frac{y(1 + j2uy - y^2)}{2u \left(1 + j \frac{y}{2u}\right)}$$

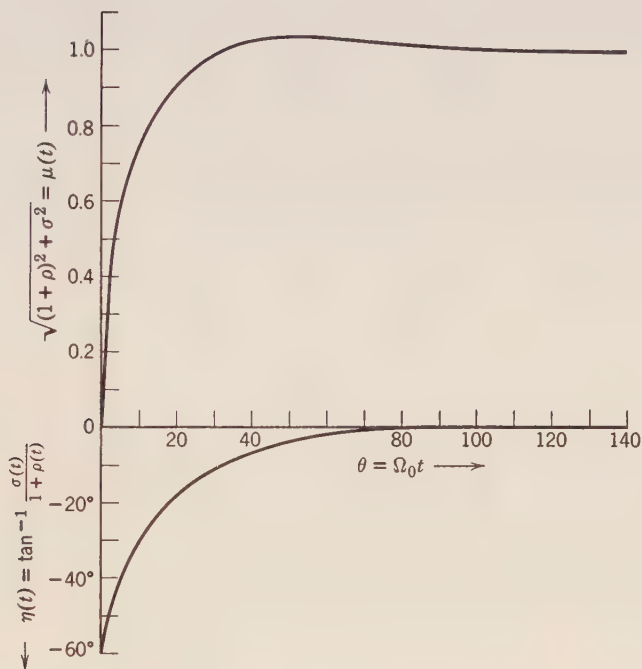


Fig. 2.19. Envelope response of circuit of Fig. 2.18.

Combining these in (107) with the other factors containing only u and y , and introducing $\theta = \Omega_0 t$, or

$$at = u\theta, \quad (\Omega - \omega_0)t = (\sqrt{1 - u^2} - y)\theta$$

we have finally

$$\rho(t) + j\sigma(t) = (a + jb)e^{-u\theta}e^{j(\sqrt{1-u^2}-y)\theta}$$

$$a + jb = \frac{y}{\sqrt{1-u^2}} \frac{(1-y^2) + j2uy}{2u + jy} \frac{u + j\sqrt{1-u^2}}{(y^2 - 1 + 2u^2) - j2u\sqrt{1-u^2}}$$

By separating real and imaginary parts we finally get the factors defining the “envelope” in (103),

$$\mu(t) = \sqrt{(1 + \rho)^2 + \sigma^2}, \quad \eta(t) = \tan^{-1} \frac{\sigma}{1 + \rho}$$

which have been plotted in Fig. 2.19. The derivative or slope of $\eta(t)$ gives the instantaneous frequency deviation from ω_0 .

2.7 Response to Frequency-Modulated Signals*

A frequency-modulated signal is generally defined in the form

$$v(t) = V_m \sin [\omega_0 t + \Phi(t)] \quad (108)$$

where ω_0 is the fixed carrier frequency and where $\Phi(t)$ is a function of time incorporating the signal to be transmitted. More specifically, if $\Phi(t) = \Phi_0$, a constant, we have the normal steady-state phase angle; if $\Phi(t)$ is a time variable, we have *angle modulation*† which can be either of the two forms

$$\Phi(t) = \Phi_0 + \mu(t) \quad \text{phase-angle modulation or phase modulation}$$

$$\Phi(t) = D \int_0^t s(t) dt \quad \text{frequency modulation}$$

In the first case we designate $\mu(t)$ as the *instantaneous phase-angle deviation*, and in the latter case we designate $(d\Phi)/(dt) = Ds(t)$ as the *instantaneous frequency* with D the maximum frequency deviation and $s(t)$ the signal, normalized so that $|s(t)| \leq 1$. For the theoretical treatment no basic difference exists between the two types and we shall concentrate upon frequency modulation as encompassing both.

The simplest type of frequency modulation is the frequency step, corresponding to the step function in amplitude change. If we formulate it in terms of (108) and the step is at $t = 0$ from a constant frequency ω_1 to another constant frequency ω_2 as in Fig. 2.20*b*, we have the deviation $D = \omega_2 - \omega_1$, the signal $s(t) = 1$, the unit step, and thus

$$\begin{aligned} \Phi(t) &= (\omega_2 - \omega_1) \int_0^t 1 dt = (\omega_2 - \omega_1)t \\ v(t) &= V_m \sin [\omega_1 t + (\omega_2 - \omega_1)t] = V_m \sin \omega_2 t \end{aligned} \quad (109)$$

* This section is based upon work carried on at the Microwave Research Institute of Polytechnic Institute of Brooklyn under Contract *N6ori-98* sponsored by the Office of Naval Research.

† Rather extensive discussions have centered upon the terminology. Since physicists usually refer to “phase” as the complete argument $[\omega_0 t + \Phi(t)]$ in (108), and engineers have always spoken of “phase angle” as meaning $\Phi(t)$ but usually implying constancy, confusion has arisen in connection with modulation terminology. See B. Van der Pol, “The Fundamental Principles of Frequency Modulation,” *J. Inst. Elec. Engrs.*, **93**, part III, 153–158 (1946).

or simply the application of the new frequency ω_2 . If we select the parallel resonant circuit of Fig. 2.20a, we are simply left with the

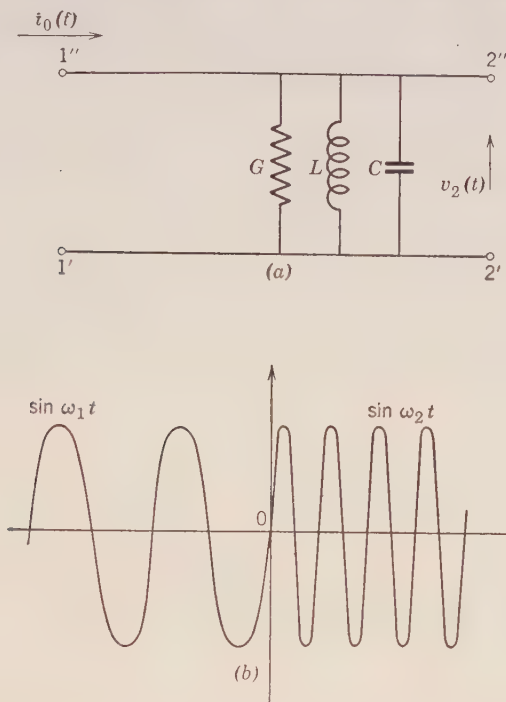


Fig. 2.20. Illustration for frequency-modulated signal; (a) parallel resonant circuit, (b) frequency step.

problem of finding the transient response to a constant frequency current source, namely

$$i_0(t) = I_m \sin \omega_2 t = \text{Im} (I_m e^{j\omega_2 t}) \quad (110)$$

The circuit represents the simple shunt admittance

$$Y(p) = G + pC + \frac{1}{pL} = \frac{C}{p} (p^2 + 2\alpha p + \Omega_0^2) \quad (111)$$

with

$$\alpha = \frac{G}{C}, \quad \Omega_0^2 = \frac{1}{LC}$$

so that we obtain for the output voltage transform in complex notation

$$\bar{V}_2(p) = \frac{I_m}{p - j\omega_2} \frac{1}{C} \frac{p}{p^2 + 2\alpha p + \Omega_0^2}$$

The inverse Laplace transform can be found by application of the a-c expansion theorem as in Table 1.4, line 13. We arrive at

$$\bar{v}_2(t) = I_m \left[\frac{e^{j\omega_2 t}}{Y(j\omega_2)} + \frac{e^{-\alpha t}}{2\Omega C} j \left(\frac{e^{j\Omega t}}{\alpha + j(\omega_2 - \Omega)} - \frac{e^{-j\Omega t}}{\alpha + j(\omega_2 + \Omega)} \right) \right] \quad (112)$$

where

$$\Omega^2 = \Omega_0^2 - \alpha^2$$

We observe that no trace is left of the previous frequency ω_1 , but that the natural frequency of the circuit itself plays a prominent part in the transient which is superimposed here although we have applied the new frequency at a zero value of the source function! If we rationalize, we get the familiar form

$$v_2(t) = \frac{I_m}{|Y(j\omega_2)|} \cos(\omega_2 t - \varphi) + \frac{I_m}{2\Omega C} e^{-\alpha t} \left[\frac{\cos(\Omega t - \varphi_1)}{[\alpha^2 + (\omega_2 - \Omega)^2]^{1/2}} - \frac{\cos(\Omega t - \varphi_2)}{[\alpha^2 + (\omega_2 + \Omega)^2]^{1/2}} \right] \quad (113)$$

with

$$\tan \varphi = \frac{2\omega_2 \alpha}{\Omega_0^2 - \omega_2^2}, \quad \tan \varphi_1 = \frac{\omega_2 - \Omega}{\alpha}, \quad \tan \varphi_2 = \frac{\omega_2 + \Omega}{\alpha}$$

No significant simplification appears possible. Even if we tune the circuit to the carrier frequency so that $\Omega_0 = \omega_0$, the instantaneous frequency, in our case ω_2 , will be different and thus prevent simpler expressions.

We might consider making the frequency variation small and slow, i.e., vary the frequency at a slow rate. It might then be assumed that the transient adjustment of the network could be disregarded, and that the circuit would remain essentially in steady state but with an admittance or impedance corresponding to the instantaneous frequency. Obviously, this can only be a first-order approximation; it is appropriately called quasi-steady-state response* and can be deduced in general form. Assume the source function to be a voltage as stated in (108) and applied to a circuit of parametric impedance $Z(p)$. To obtain simpler expressions, let us write the source voltage in complex notation

$$\bar{v}(t) = V_m \exp [j\omega_0 t + j\Phi(t)] \quad (114)$$

* J. R. Carson and T. C. Fry, "Variable Frequency Circuit Theory," *Bell System Tech. J.*, **16**, 513-540 (1937).

with the Laplace transform

$$\bar{V}(p) = V_m \mathfrak{L}(e^{j\omega_0 t} e^{j\Phi(t)}) = V_m M(p - j\omega_0) \quad (115)$$

if we make use of Table 1.4, line 5, and define

$$\mathfrak{L}e^{j\Phi(t)} = M(p) \quad (116)$$

The Laplace transform of the current follows as

$$\bar{I}(p) = \frac{\bar{V}(p)}{Z(p)} = V_m M(p - j\omega_0) \sum_{\alpha=1}^n \frac{B_\alpha}{p - p_\alpha} \quad (117)$$

where we applied the expansion into partial fractions to the impedance, assuming it to be a rational function of p of degree n . The inverse Laplace transform is found best by application of the Borel theorem from Table 1.5, line 1; we define

$$\begin{aligned} F_1(p) &= \frac{1}{p - p_\alpha} & f_1(t) &= e^{p_\alpha t} \\ F_2(p) &= M(p - j\omega_0) & f_2(t) &= e^{j\omega_0 t} e^{j\Phi(t)} \end{aligned}$$

so that we have for the product

$$e^{p_\alpha t} \int_0^t e^{-p_\alpha \tau} e^{j\omega_0 \tau} e^{j\Phi(\tau)} d\tau = e^{p_\alpha t} \int_{\tau=0}^{\tau=t} e^y \frac{dy}{y'}$$

Here we have introduced

$$y = (-p_\alpha + j\omega_0)\tau + j\Phi(\tau), \quad y' = (-p_\alpha + j\omega_0) + j \frac{d\Phi}{d\tau}$$

Integrating by parts and selecting

$$du = e^y dy, \quad u = e^y; \quad v = \frac{1}{y'}, \quad dv = \frac{-y''}{(y')^2}$$

we obtain further

$$e^{p_\alpha t} \left(\frac{e^y}{y'} + \int \frac{y''}{(y')^2} e^y dy \right)_{\tau=0}^t \quad (118)$$

where $y'' = j[(d^2\Phi)/(d\tau^2)]$ might be assumed small enough so that this integral might be disregarded. This assumption will be justified only if, in $\Phi(t)$, defined as

$$\Phi(t) = D \int_0^t s(t) dt, \quad \frac{d\Phi}{dt} = Ds(t), \quad \frac{d^2\Phi}{dt^2} = Ds'(t) \quad (119)$$

the maximum frequency deviation D is small compared with the carrier frequency ω_0 , and the time variation of $s(t)$ is slow. If these conditions are met, then (118) reduces to the first term which gives in the limits

$$\frac{e^{p_\alpha t}}{(-p_\alpha + j\omega_0) + j \frac{d\Phi}{dt}} \{ \exp [-p_\alpha t + j\omega_0 t + j\Phi(t)] - 1 \}$$

Introducing this back into the inverse transform of (117), we obtain for the sums in complex notation

$$\bar{i}(t) = V_m \sum_{(\alpha)} \frac{B_\alpha}{-p_\alpha + j\omega_i} \exp [j\omega_0 t + j\Phi(t)] - V_m \sum_{(\alpha)} \frac{B_\alpha e^{p_\alpha t}}{-p_\alpha + j\omega_i} \quad (120)$$

The second sum constitutes part of the transient starting at $t = 0$ and decaying with the normal time constants of the network; it is not characteristic of the frequency-modulated signal at all, and could be evaluated in complete form much simpler as the decay of the initial energy in the network.

The first sum, however, is a new result. The exponential function, not containing any network parameters, but indeed being the applied voltage from (112), can be taken before the sum, which reads by comparison with (117),

$$\sum_{(\alpha)} \frac{B_\alpha}{j\omega_i - p_\alpha} = \frac{1}{Z(j\omega_i)}$$

and which is nothing else but the instantaneous frequency impedance! We thus have the final form

$$\bar{i}(t) = \frac{V_m \exp [j\omega_0 t + j\Phi(t)]}{Z(j\omega_i)} \quad (121)$$

the quasi-steady-state solution, where

$$\omega_i = \omega_0 + \frac{d\Phi}{dt} = \omega_0 + Ds(t)$$

contains directly the desired signal. In rationalized form

$$i(t) = \frac{V_m}{|Z(j\omega_i)|} \sin [\omega_0 t + \Phi(t) - \varphi(\omega_i)] \quad (122)$$

where it is obvious that the current carries amplitude modulation through the absolute value of the variable impedance and where the

signal $\Phi(t)$ has superimposed a distortion term $\varphi(\omega_i)$ coming from the phase characteristic of the network impedance. The corresponding form for the current source with frequency modulation is obtained by the sequential substitution of v , I_m , Y for the quantities i , V_m , Z . The same result (122), though by different methods, was derived by Carson and Fry and by Van der Pol.*

For numerical computations it is necessary to stipulate the function $\Phi(t)$ or its derivative $Ds(t)$, the instantaneous frequency deviation. The simplest form of sustained modulation is, of course, the sinusoidal one, for which

$$s = \cos \mu t, \quad \Phi(t) = + \frac{D}{\mu} \sin \mu t = +m \sin \mu t$$

where m is called the *modulation index*. Several different methods have been used to obtain the steady-state response in this case which is also amenable to Fourier series analysis. Actually, the sinusoidal frequency modulation and its spectral resolution is most frequently included in the steady-state analysis of network response.† A very comprehensive treatment for the evaluation of circuit responses is given by Bloch.‡ If we expand the network function as a function of $p + j\omega$ into a complex Fourier series in terms of ω in accordance with Vol. 1, (6.41), choosing as fundamental period Ω the range of frequencies ω of interest in the problem (covering the significant sidebands of the signal function), we obtain

$$\frac{1}{Z(j\omega)} = \sum_{n=-N}^{+N} \bar{C}_n e^{j \frac{2\pi n}{\Omega} \omega}$$

with the coefficients

$$\bar{C}_n = \frac{1}{\Omega} \int_{-\Omega/2}^{+\Omega/2} \frac{1}{Z(j\omega)} e^{-j \frac{2\pi n}{\Omega} \omega} d\omega$$

This is an alternative to the partial fraction expansion and can be used in (117). Instead of the Laplace transform it is now better to use the complex Fourier integral method or Fourier transforms (see section

* Carson and Fry, *loc. cit.*; Van der Pol, *loc. cit.* Further expansions for practical calculations are given by A. S. Gladwin, "The Distortion of Frequency-Modulated Waves by Transmission Methods," *Proc. I.R.E.*, **35**, 1436-1445 (1947).

† See Cherry,^{A4} chapter 2. See also S. Goldman, *Frequency Analysis, Modulation and Noise*, McGraw-Hill, New York, 1948; A. Hund, *Frequency Modulation*, McGraw-Hill, New York, 1942.

‡ A. Bloch, "Modulation Theory," *J. Inst. Elec. Engrs.*, **91**, part III, 31 (March 1944).

1.3). The frequency-modulated input voltage (114) has as Fourier transform

$$\bar{V}(\omega) = \mathfrak{F}\bar{v}(t)$$

so that the Fourier transform of the current response becomes

$$\bar{I}(\omega) = \frac{\bar{V}(\omega)}{Z(j\omega)} = \sum_{n=-N}^{+N} \bar{C}_n e^{j\frac{2\pi n}{\Omega}\omega} \bar{V}(\omega)$$

The inverse Fourier transforms can be found by using the “delay” relation in the Table 1.2, line 2, where for each exponential

$$\mathfrak{F}^{-1}\bar{V}(\omega)e^{\pm j\frac{2\pi n}{\Omega}\omega} = \bar{v}\left(t \pm \frac{2\pi n}{\Omega}\right)$$

so that the complete response becomes in complex notation

$$\bar{i}(t) = \sum_{n=-N}^{+N} \bar{C}_n \bar{v}\left(t + \frac{2\pi n}{\Omega}\right) \quad (123)$$

This method also avoids finding explicitly the transform of the signal voltage which it is difficult to obtain, and substitutes the evaluation of the coefficients \bar{C}_n , which might not be an easy task except with simple functions $Z(j\omega)$. It is called the method of *paired echos* and was introduced by Wheeler* for the evaluation of transients involving amplitude-modulated signals. Its extension to frequency-modulated signals was made by Bloch, *loc. cit.*, in terms of the complex Fourier series, and by Frantz† in terms of real Fourier series for each real and imaginary component of $Z(j\omega)$. A still different expansion was used by Gold‡ who developed the network function into a series of orthonormal polynomial functions.

All these expansions have great value in specific computations but do not permit a closed form representation such as the quasi-steady-state expression (122) where this is applicable. Let us use it on the same parallel resonant circuit to which we applied the frequency step, but now with sinusoidal frequency modulation. Since we have a cur-

* H. Wheeler, “The Interpretation of Amplitude and Phase Distortion in Terms of Paired Echos,” *Proc. I.R.E.*, **27**, 359–384 (1939). See also Cherry,^{A4} section 74; and R. V. L. Hartley, “A More Symmetrical Fourier Analysis Applied to Transmission Problems,” *Proc. I.R.E.*, **30**, 144–150 (1942).

† W. J. Frantz, “The Transmission of a Frequency-Modulated Wave through a Network,” *Proc. I.R.E.*, **34**, 114–125 (1946).

‡ B. Gold, “The Solution of Steady State Problems in FM,” *Proc. I.R.E.*, **37**, 1264–1269 (1949).

rent source, we shall use the dual form to (122), namely

$$v(t) = \frac{I_m}{|Y(j\omega_i)|} \sin [\omega_0 t + \Phi(t) - \varphi(\omega_i)] \quad (124)$$

The admittance from (111) can be expanded into the partial fractions

$$\frac{1}{Y(p)} = \frac{1}{C(p_1 - p_2)} \left(\frac{p_1}{p - p_1} - \frac{p_2}{p - p_2} \right)$$

with the roots

$$p_{1,2} = -\alpha \pm j\Omega, \quad \Omega = \sqrt{\Omega_0^2 - \alpha^2} \cong \Omega_0$$

We need to introduce $p = j\omega_i$, with ω_i the instantaneous frequency, which for sinusoidal modulation is

$$\begin{aligned} \omega_i &= \omega_0 + Ds(t) = \omega_0 + D \cos \mu t \\ \Phi(t) &= D \int s(t) dt = \frac{D}{\mu} \sin \mu t \end{aligned} \quad (125)$$

where μ is the modulating frequency assumed low compared with the carrier frequency ω_0 . We have, therefore

$$\frac{1}{Y(j\omega_i)} = \frac{1}{2j\Omega C} \left(\frac{-\alpha + j\Omega}{j\omega_i + \alpha - j\Omega} - \frac{-\alpha - j\Omega}{j\omega_i + \alpha + j\Omega} \right)$$

We might further assume that the circuit is tuned to the carrier frequency so that $\Omega = \omega_0 \cong \Omega_0$. The second term in parenthesis will contain the sum $\omega_0 + \Omega = 2\omega_0$ in the denominator which will make it small compared with the first term, so we shall disregard it. This permits us to write

$$\frac{1}{Y(j\omega_i)} = \frac{1}{2\omega_0 C} \sqrt{\frac{\omega_0^2 + \alpha^2}{(Ds)^2 + \alpha^2}} \left(\tan^{-1} \frac{\alpha}{\omega_0} - \tan^{-1} \frac{Ds}{\alpha} \right)$$

and therefore for the voltage in (124)

$$\begin{aligned} v(t) &= \frac{I_m}{2\omega_0 C} \left(\frac{\omega_0^2 + \alpha^2}{D^2 \cos^2 \mu t + \alpha^2} \right)^{1/2} \sin \left[\omega_0 t + \frac{D}{\mu} \sin \mu t \right. \\ &\quad \left. - \tan^{-1} \left(\frac{D}{\alpha} \cos \mu t \right) + \tan^{-1} \frac{\alpha}{\omega_0} \right] \end{aligned} \quad (126)$$

We observe the variation of amplitude with frequency μ and the distortion term in the phase of the output voltage. The instantaneous frequency in the output is now the time derivative of the whole argu-

ment of the sine function, which is

$$\omega_i' = \omega_0 + D \cos \mu t + D \frac{\alpha \mu \sin \mu t}{\alpha^2 + D^2 \cos^2 \mu t} \quad (127)$$

To make the distortion* small, we must satisfy both conditions

$$D/\alpha < 1, \quad \mu/\alpha < 1$$

where we can interpret 2α as the bandwidth of the tuned circuit in the conventional definition. Actual comparative computations† indicate that the quasi-steady-state approximation in the above example can be justified for $(D\mu)/\alpha^2 < 0.3$.

Under the same conditions, we can introduce into the expression for the instantaneous frequency admittance

$$\tan^{-1} \frac{Ds}{\alpha} \approx \frac{Ds}{\alpha}$$

and

$$\sqrt{\alpha^2 + (Ds)^2} \approx \alpha \left[1 + \frac{1}{2} \left(\frac{Ds}{\alpha} \right)^2 \right]$$

so that the general quasi-steady-state response for any small signal becomes

$$v(t) = V_0 \left[1 - \frac{1}{2} \left(\frac{Ds}{\alpha} \right)^2 \right] \sin \left(\omega_i t - \frac{D}{\alpha} s(t) + \tan^{-1} \frac{\alpha}{\omega_0} \right) \quad (128)$$

In this form, the spurious amplitude modulation is seen to be at twice the signal frequency and can readily be gauged from the explicit form of the signal itself. Ideal limiter and discriminator action will recover the signal from ω_i in accordance with (125) and leave as distortion

$$- \frac{D}{\alpha} \frac{d}{dt} s(t)$$

For a sawtooth variation of frequency this means a periodic slight displacement of the signal and slight jumps at the half periods.

PROBLEMS

2.1 Deduce the standard fourpole parameters for the circuit in Fig. 2.21 by matrix methods.

* Extensive computations and comparison with measurements are given by D. L. Jaffe, "A Theoretical and Experimental Investigation of Tuned-Circuit Distortion in Frequency-Modulated Systems," *Proc. I.R.E.*, **33**, 318-334 (1945).

† Unpublished report by J. Brogan, Microwave Research Institute of Polytechnic Institute of Brooklyn, 1948.

2.2 Apply a square-wave voltage pulse $v_1(t)$ to the network in Fig. 2.21 and find the output voltage $v_2(t)$. Compare the signal shapes.

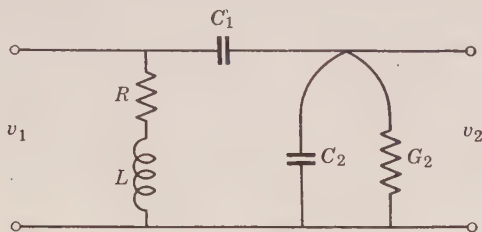


Fig. 2.21. Video amplifier interstage network.

2.3 Apply a single linear sawtooth voltage $v_1(t) = V_m t/T$ to the network in Fig. 2.21. What should be the value of inductance L to give nearly linear rise of the voltage $v_2(t)$?

2.4 A train of three sawtooth voltage pulses as in Fig. 1.17a is applied to the balanced fourpole in Fig. 2.9. (a) Find the output voltage pulses (use the superposition principle). (b) What should be the value of Ω_c to give distinct separation of these three pulses?

2.5 A current impulse $i_1(t) = M_i S_0(t)$ is applied to the network in Fig. 2.21. Find the output voltage $v_2(t)$.

2.6 Obtain the standard fourpole parameters for the T-section in Fig. 2.4e if $Z_1 = Z_2 = pL + 1/(pC)$, $Y = 2pC$. Apply a step voltage $V_m I$ at the input terminal pair and find the output voltage with resistive termination.

2.7 In the circuit of problem 2.6 discuss the steady-state characteristics and relate resonant frequencies to filter behavior.

2.8 Obtain the standard fourpole parameters for the network in Fig. 2.22.

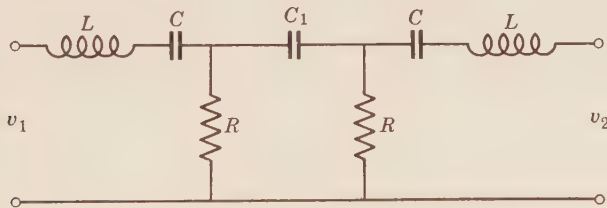


Fig. 2.22. Symmetrical cascade of fourpoles for problems 2.8 and 2.9.

2.9 Apply a step voltage $V_m I$ to the network in Fig. 2.22 (a) when the output terminals are open-circuited; (b) when the output terminals are short-circuited.

2.10 Apply a step voltage $V_m I$ to the circuit in Fig. 1.19 and find the output voltage. Do the same for two and three identical sections in cascade. Use the standard fourpole parameter notation.

2.11 In the lattice in Fig. 2.7b choose $P = Q = pL + 1/(pC)$, $S = T = R$ and terminate the output into resistance R_1 . (a) Apply a unit step voltage and find the output voltage $v_2(t)$. (b) Discuss the steady-state characteristics of the network.

2.12 A cascade arrangement of three sections of series C and shunt R is supplied over a series resistance R from a voltage source. If a step voltage $V_m I$ is

applied, find the output voltage. Use (a) Kirchhoff equations, (b) the matrix method.

2.13 A transformer with small losses has a pure capacitance as load. Find the secondary current if the alternating voltage at the primary terminals is suddenly switched on at $t = 0$.

2.14 A transformer has carried rated load under steady-state conditions. At $t = 0$, a short circuit occurs at the secondary terminals. Find the input current.

2.15 A power transformer is operating under steady-state a-c conditions into a resistive load (presented by the surge resistance of the transmission line). At $t = 0$ an impulse voltage wave $v_2(t) = M_v S_0(t)$ appears across the resistive load (having come over the transmission line). Find the current response both in the secondary and primary winding of the transformer.

2.16 Construct the table of natural frequencies for the tuned coupled circuits of Fig. 2.16*b* as the number of stages is increased from one to two and to three, in accordance with relation (83) for the output voltage under open-circuited conditions.

2.17 Find the response of the coupled circuit of Fig. 2.16*a* for a single linearly rising sawtooth voltage of duration T as in Fig. 1.6, if the secondary terminal pair (a) is open-circuited, (b) feeds a purely resistive load. We can assume k^2 to be considerably smaller than unity.

2.18 Plot the complex voltage transfer function of the coupled circuits in Fig. 2.16*a* for open-circuited secondary terminals. Select $k^2 = 0.2$ and $L_1 C_1 = m L_2 C_2$, with $m = 0.8, 1, 1.2$.

2.19 Find the envelope response of the circuit in Fig. 2.18 for a linearly rising amplitude $m(t) = Mt/T$ for $t \leq T$ and zero afterwards. (a) Evaluate the response for $Q = 50$ and $\omega_0 = 0.98\Omega_0$. We assume $\omega_0 T$ very large. (b) Compare the result with the response to the sawtooth pulse without carrier.

2.20 Find the envelope response of the circuit in Fig. 2.5*b* for the step function modulated sinusoidal carrier current $I_0 \sin \omega_0 t$. Assume the carrier frequency $\omega_0 = \frac{1}{2}\Omega_c$.

2.21 Find the quasi-steady-state response of the circuit in Fig. 2.18 for sinusoidally frequency-modulated input current $\bar{i}(t) = I_m \exp(j\omega_0 t + jm \sin \mu t)$ where m is the modulation index.

2.22 Compare the method of paired echos (see section 2.7), the method of Gold, *loc. cit.*, and the quasi-steady-state solution for the circuit in Fig. 2.20 with sinusoidally frequency-modulated input current which is treated in the text of section 2.7. For computations select the values $D/\alpha = 0.4$, $\mu/\alpha = 0.4$, $D/\omega_0 = 0.02$, $\omega_0 = \Omega$.

2.23 Apply the frequency-modulated voltage $v_1(t) = V_m \sin[\omega_0 t + \Phi(t)]$ to the magnetically coupled circuits in Fig. 2.16*a*. Make the assumptions required for the validity of the quasi-steady-state solution and obtain it for the output voltage (a) in general form, (b) for sinusoidal frequency modulation.

2.24 Carry the development of (117) through (120) to the next term, assuming that y'' must be retained whereas y''' could be considered negligibly small. Compare the expanded solution with the quasi-steady-state solution.

3. PASSIVE FOURPOLE LINES (WAVE FILTERS)

The cascade arrangement of fourpoles in any repetitive pattern constitutes a *fourpole line*; the group of elements or fourpoles which define the individual repetitive unit will be considered as a single fourpole, characterized by the general fourpole parameters

$$\begin{bmatrix} \alpha & \beta \\ \epsilon & \mathfrak{D} \end{bmatrix} \text{ unsymmetrical, } \begin{bmatrix} \alpha & \beta \\ \epsilon & \alpha \end{bmatrix} \text{ symmetrical}$$

The general theory of the fourpole line can be carried to a formal solution so that the detailed values of these general parameters need only be specified when quantitative results are desired.

A very large field of practical applications has been developed in all branches of engineering and applied physics. Perhaps the earliest applications occurred in mechanical systems where beams with periodic supports, crank shafts with torques applied by equally spaced cylinder pistons, or vibrating strings with periodically spaced beads led to problems involving repetitive structures.* The simulation of electrical transmission lines by lumped-parameter systems, the so-called artificial lines,† introduced the subject into electrical engineering. However, the greatest contribution, without doubt, came with the invention of electric *wave filters*,‡ which are fourpole lines of specific selective properties of transmission, and which in turn stimulated far-reaching developments in acoustical and mechanical filters.§

* Thomson,^{D18} chapter 7; T. V. Karman and M. A. Biot, *Mathematical Methods in Engineering*, chapter 5, McGraw-Hill, New York, 1940.

† A. E. Kennelly, *Artificial Lines and Nets*, Wiley, New York, 1912.

‡ G. A. Campbell, "Physical Theory of the Electric Wave-Filter," *Bell System Tech. J.*, **1**, 1-32 (1922); K. W. Wagner, "The Theory of Wave Filters with Applications" [German], *Arch. Elektrotech.*, **3**, 315 (1915); "Coil and Condenser Transmission Systems" [German], *Arch. Elektrotech.*, **8**, 61 (1919).

§ P. M. Morse, *Vibration and Sound*, McGraw-Hill, New York, 1936; H. F. Olson, *Elements of Acoustical Engineering*, Van Nostrand, New York, 1940. G. W. Stewart and R. B. Lindsay, *Acoustics*, Van Nostrand, New York, 1930. R. B. Lindsay, "The Filtration of Sound," *J. Appl. Phys.*, **9**, 612-22 (1938), **10**, 680-87 (1939).

3.1 The General Solution

As a general arrangement we take N like fourpoles in cascade shown schematically in Fig. 3.1, with a terminating two-terminal network at the far end and with either a voltage or current source supplying energy at the near end over a terminal impedance Z_S (admittance Y_S). We shall number the fourpoles from zero to $(N - 1)$ as a matter of convenience in stating the terminal conditions; the final solution will not, of course, depend upon this selection and will be the same for a numbering system 1 to N . Each fourpole is characterized by the general parameters $(\alpha, \beta, \mathcal{C}, \mathcal{D})$ and the Laplace transforms of its

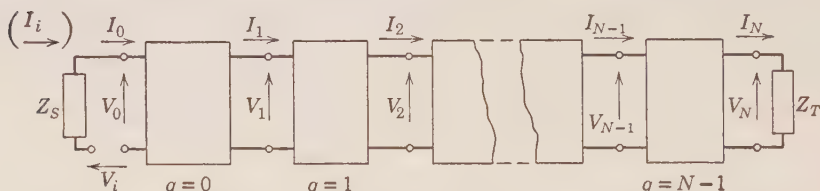


Fig. 3.1. General fourpole line of N like fourpoles with terminal impedances.

input and output quantities are related by the linear equations (2.11) or, in matrix form, (2.12), to wit

$$\begin{aligned} V_q(p) &= \alpha V_{q+1}(p) + \beta I_{q+1}(p) \\ I_q(p) &= \mathcal{C} V_{q+1}(p) + \mathcal{D} I_{q+1}(p) \end{aligned} \quad (1)$$

In accordance with Fig. 3.1 this system holds for the q th fourpole in the chain and we could write the respective relations for all the fourpoles from $q = 0$ to $q = N - 1$, and eliminate all the current and voltage transforms except V_0, I_0, V_N, I_N either by direct elimination, by determinant solution, or by the matrix process indicated at the end of section 2.3 and actually carried through for up to four symmetrical fourpoles. For the remaining transforms we must find again a linear system

$$\begin{aligned} V_0(p) &= \alpha_L V_N(p) + \beta_L I_N(p) \\ I_0(p) &= \mathcal{C}_L V_N(p) + \mathcal{D}_L I_N(p) \end{aligned} \quad (2)$$

where $\alpha_L, \beta_L, \mathcal{C}_L, \mathcal{D}_L$ are the rather complex over-all line parameters obtained, e.g., from the N -fold matrix product

$$\begin{bmatrix} \alpha_L & \beta_L \\ \mathcal{C}_L & \mathcal{D}_L \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \mathcal{C} & \mathcal{D} \end{bmatrix}^N \quad (3)$$

Together with the final two mesh equations which introduce the terminal impedances,

Far end (receiving): $V_N(p) = Z_T I_N(p)$

Near end (sending): $V_0(p) = V_i(p) - Z_S I_0(p)$, voltage source
 $I_0(p) = I_i(p) - Y_S V_0(p)$, current source (4)

we have now four equations left, sufficient to determine sending and receiving end quantities. We observe that if we rewrite the near end terminal condition applying to a current source in the same form as that applying to a voltage source, namely

$$V_0(p) = \frac{I_i(p)}{Y_S} - \frac{I_0(p)}{Y_S} = Z_S I_i(p) - Z_S I_0(p)$$

we have the correspondence

$$V_i(p) \rightarrow Z_S I_i(p) = \frac{I_i(p)}{Y_S} \quad (4a)$$

so that we can make ready substitution where desired. It should be noted that the $\alpha_L, \beta_L, \mathcal{C}_L, \mathcal{D}_L$ must be fractions of polynomials as long as N is finite since we have restricted our treatment to passive fourpole lines. Though quite complicated, the further solution is straightforward and feasible with the methods developed so far.

But what we have done is nothing more than ordinary network analysis aided by the matrix notation of fourpole theory. A much simpler and more powerful approach is possible. Suppose we try to establish a relationship between currents alone, or between voltages alone. For example, repeating the second line of (1) for fourpole $(q-1)$,

$$I_{q-1}(p) = \mathcal{C}V_q(p) + \mathcal{D}I_q(p)$$

we can take $V_q(p)$ from here, $V_{q+1}(p)$ from the second line of (1), introduce these into the first line of (1) and arrive at

$$I_{q-1}(p) - (\alpha + \mathcal{D})I_q(p) + I_{q+1}(p) = 0 \quad (5)$$

where we used the passive fourpole characteristic from (2.19)

$$\alpha\mathcal{D} - \beta\mathcal{C} = 1 \quad (6)$$

Quite similarly, it is possible to deduce the identical form for the voltages

$$V_{q-1}(p) - (\alpha + \mathcal{D})V_q(p) + V_{q+1}(p) = 0 \quad (7)$$

Both equations relate the transforms of three successive fourpoles, making it appear that the fourpole number will enter the solution as a parameter that can only take real integral values. We know on the other hand, however, that we could evaluate these transforms as regular polynomial fractions if we followed the conventional network solution outlined above, keeping then q , $q - 1$, etc., attached to the transforms just as indices. We conclude, therefore, that the seeming appearance of indices as a variable is not in the form of a new independent variable necessitating a restudy of Laplace transform theory, but merely a parameter not affecting the existence and convergence of the transforms any more than other real parameters.*

In terms of q , both (5) and (7) represent the aspect of *linear difference equations*, relating functions of p such that the parameter q does not appear in the coefficients. These are very similar to the linear differential equations with constant coefficients and at once invite an attempt to try exponential functions as a solution, e.g., $e^{q\Gamma}$ where Γ is a nondimensional quantity which must be determined from the difference equations. If we introduce this exponential into (5) with appropriate adjustment of the parameter q , we have

$$e^{(q-1)\Gamma} - (\alpha + \mathfrak{D})e^{q\Gamma} + e^{(q+1)\Gamma} = 0 \quad (8)$$

and, as we suspected, it reduces to a characteristic equation for Γ

$$e^{-\Gamma} - (\alpha + \mathfrak{D}) + e^{+\Gamma} = 0$$

or better

$$\cosh \Gamma = \frac{\alpha + \mathfrak{D}}{2}, \quad \Gamma = \ln \left[\frac{\alpha + \mathfrak{D}}{2} + \sqrt{\left(\frac{\alpha + \mathfrak{D}}{2}\right)^2 - 1} \right] \quad (9)$$

Since $\cosh \Gamma$ is an even function, we have to admit $\pm \Gamma$ as two independent values; in analogy to the differential equations, (5) and (7) are difference equations of second order.

Instead of an exponential, we could also have chosen an algebraic solution,[†] say y^q : then (5) would read

$$y^{q-1} - (\alpha + \mathfrak{D})y^q + y^{q+1} = 0$$

* A rather thorough study of the applicability of Laplace transforms to difference equations is presented by Gardner-Barnes,^{D8} chapter IX, who use the class of *jump functions*, which confirms the direct attack on the solution presented here.

† G. Boole, *Treatise on the Calculus of Finite Differences*, 3rd edition, G. Stechert, New York, 1931; N. E. Nörlund, *Differenzenrechnung*, J. Springer, Berlin, 1924; L. M. Milne-Thomson, *The Calculus of Finite Differences*, Macmillan, London, 1933. See also brief treatments in Karman-Biot, *op. cit.*, chapter XI; Widder,^{D24} pp. 416-419.

which reduces to

$$y^2 - (\mathfrak{A} + \mathfrak{D})y + 1 = 0$$

with the roots

$$y_{1,2} = \frac{\mathfrak{A} + \mathfrak{D}}{2} \pm \sqrt{\left(\frac{\mathfrak{A} + \mathfrak{D}}{2}\right)^2 - 1} \quad (10)$$

If we make the substitution $y = e^\Gamma$ we observe identity of (9) and (10), and the \pm sign of the square root corresponds to $\pm\Gamma$ values in (9).

The general solution of the second-order difference equation (5) must be of the type

$$I_q(p) = Me^{-q\Gamma} + We^{+q\Gamma} \quad (11)$$

where M and W are integration constants to be determined from the terminal conditions (4). Since these involve also the voltage transforms, we need to find the general solution of $V_q(p)$, which we write, replacing $(q + 1)$ by q in the second line of (1)

$$V_q(p) = \frac{1}{\mathfrak{C}} I_{q-1}(p) - \frac{\mathfrak{D}}{\mathfrak{C}} I_q(p)$$

and with (11)

$$\begin{aligned} V_q(p) &= \frac{M}{\mathfrak{C}} (e^\Gamma - \mathfrak{D})e^{-q\Gamma} - \frac{W}{\mathfrak{C}} (\mathfrak{D} - e^{-\Gamma})e^{+q\Gamma} \\ &= M'e^{-q\Gamma} - W'e^{+q\Gamma} \end{aligned} \quad (12)$$

The terminal conditions are given by (4) or, if we introduce (11) and (12) which are valid for any value $0 \leq q \leq N$, by

$$\begin{aligned} V_0(p) &= M' - W' = V_i(p) - Z_0(M + W) \\ V_N(p) &= M'e^{-N\Gamma} - W'e^{+N\Gamma} = Z_t(Me^{-N\Gamma} + We^{+N\Gamma}) \end{aligned} \quad (13)$$

These last relations permit the evaluation of M and W .

In order to simplify the final solution, we shall define from (12) with the aid of (9)

$$\begin{aligned} \frac{M'}{M} = Z_{c1} &= \frac{1}{\mathfrak{C}} \left[+ \frac{\mathfrak{A} - \mathfrak{D}}{2} + \sqrt{\left(\frac{\mathfrak{A} + \mathfrak{D}}{2}\right)^2 - 1} \right] \\ \frac{W'}{W} = Z_{c2} &= \frac{1}{\mathfrak{C}} \left[- \frac{\mathfrak{A} - \mathfrak{D}}{2} + \sqrt{\left(\frac{\mathfrak{A} + \mathfrak{D}}{2}\right)^2 - 1} \right] \end{aligned} \quad (14)$$

which we designate as *iterative impedances*. We then obtain from (13) with (14)

$$M(Z_S + Z_{C1}) + W(Z_S - Z_{C2}) = V_i(p)$$

$$Me^{-N\Gamma}(Z_T - Z_{C1}) + We^{N\Gamma}(Z_T + Z_{C2}) = 0$$

Let us finally use *reflection coefficients* as follows

$$\rho_S = \frac{Z_S - Z_{C2}}{Z_S + Z_{C1}}, \quad \rho_T = \frac{Z_T - Z_{C1}}{Z_T + Z_{C2}} \quad (15)$$

then the solutions for the integration constants M and W become

$$M = \frac{e^{N\Gamma}}{\Delta} \frac{V_i(p)}{Z_S + Z_{C1}}, \quad W = \frac{-\rho_T e^{-N\Gamma}}{\Delta} \frac{V_i(p)}{Z_S + Z_{C1}}$$

with

$$\Delta = e^{N\Gamma} - \rho_S \rho_T e^{-N\Gamma}$$

Thus, we obtain the general solutions for any voltage and current transform pair

$$V_q(p) = \frac{V_i(p)}{Z_{C1} + Z_S} \frac{Z_{C1} e^{(N-q)\Gamma} + Z_{C2} \rho_T e^{-(N-q)\Gamma}}{e^{N\Gamma} - \rho_S \rho_T e^{-N\Gamma}} \quad (16)$$

$$I_q(p) = \frac{V_i(p)}{Z_{C1} + Z_S} \frac{e^{(N-p)\Gamma} - \rho_T e^{-(N-p)\Gamma}}{e^{N\Gamma} - \rho_S \rho_T e^{-N\Gamma}} \quad (17)$$

If we reflect upon the nature of Γ , Z_{C1} , and Z_{C2} and their dependence respectively indicated in (9) and (14) upon the general fourpole parameters (\mathfrak{A} , \mathfrak{B} , \mathfrak{C} , \mathfrak{D}), we appreciate at once that this general transform solution has not been translated into time functions by means of the inverse Laplace transforms. Several special cases have been solved, however, and we hasten to deduce these from the formidable general forms, attractive as these might be as the "formal solutions" of all finite fourpole line problems.

3.2 Discussion of the General Solution; Tractable Cases

It appears that we can make the most radical reduction of the general forms by choosing $Z_T = Z_{C1}$, because this makes $\rho_T = 0$, and thus leads to

$$\left. \begin{aligned} V_q(p) &= \frac{Z_{C1}}{Z_{C1} + Z_S} V_i(p) e^{-q\Gamma} \\ I_q(p) &= \frac{1}{Z_{C1} + Z_S} V_i(p) e^{-q\Gamma} \end{aligned} \right\} \text{for } Z_T = Z_{C1}, \rho_T = 0 \quad (18)$$

Since N has dropped out, this solution is obviously valid for the *infinite line*; we can also interpret this to mean that the termination

$Z_T = Z_{C1}$ is a matched one, suppressing any reflections from the far end $q = N$. To demonstrate this, let $N \rightarrow \infty$ in (16) and (17). Assuming that $\text{Re } \Gamma > 0$ [we just need to select as $+\Gamma$ that solution (10) for which this is true], we can drop the second terms in the numerators and denominators because of the negative exponentials and arrive at once at (18). For this reason, as well as the fact that from (18)

$$Z_{C1} = \frac{V_q(p)}{I_q(p)} \quad \text{for } \rho_T = 0$$

we call Z_{C1} the *iterative characteristic impedance* for transmission towards the far end, or *to the right* if we refer to Fig. 3.1 as drawn. Unfortunately, though (18) looks simple, its inverse transform is not easily obtainable and we shall defer it to section 3.7.

The exponent Γ appears in (18) as a measure of voltage or current change in parametric form as we progress from one section to the next. Indeed, the ratios

$$\frac{V_q(p)}{V_{q-1}(p)} = \frac{I_q(p)}{I_{q-1}(p)} = e^{-\Gamma} \quad (19)$$

can be taken as a definition of Γ in the sense

$$\Gamma = \ln \frac{V_{q-1}}{V_q} = \ln \frac{I_{q-1}}{I_q}$$

For a single frequency under steady-state conditions, for which we can set $p = j\omega$, Γ represents a measure of the *transmission*; it is designated as the *propagation constant*, its real value being the *attenuation constant* and its imaginary part being the *phase constant*

$$\Gamma(j\omega) = \alpha(\omega) + j\beta(\omega) \quad (19a)$$

If we take the frequency as an independent variable, as we shall do in general, then Γ is more appropriately called the *propagation function* (of frequency), and correspondingly α is the *attenuation function* and β the *phase function*. Obviously, (19) could be also considered as a *characteristic transfer function* for the symmetrical fourpole line, forming thus a firm link between the lumped-network concept and the propagation aspects of an infinitely extended line.

Since we have unsymmetrical fourpoles in this general case, we can turn all of them around, which means interchanging \mathfrak{A} and \mathfrak{D} as discussed at the beginning of section 2.2. This interchange does not affect Γ at all in (9), but it does interchange Z_{C1} and Z_{C2} in (14). If we transfer this interchange into ρ_S and ρ_T in (15) we see that $\rho_T = 0$

for $Z_T = Z_{C2}$, or that now Z_{C2} has taken over the role of Z_{C1} above. Therefore we also call Z_{C2} the *iterative characteristic impedance* but for transmission towards the sending end or *to the left* if we leave the fourpoles in Fig. 3.1 untouched and interchange instead the two terminations including the source.

Of considerable practical interest is, fortunately, the *symmetrical fourpole line* for which $\mathfrak{A} = \mathfrak{D}$. We have from (9)

$$\cosh \Gamma = \mathfrak{A} = \mathfrak{D} \quad (20)$$

and from (14)

$$Z_{C1} = Z_{C2} = Z_C = \sqrt{\frac{\mathfrak{B}}{\mathfrak{C}}} \quad (21)$$

a single *characteristic impedance* of the fourpole line. No further simplification can be made, however, unless we again terminate at the far end into the characteristic impedance $Z_T = Z_C$, in which case we again have (18), the transform solution for the infinite line, or for the finite, perfectly matched line. If we also match at the sending end, i.e., make $Z_S = Z_C$, then

$$V_q(p) = \frac{1}{2} V_i(p) e^{-q\Gamma}, \quad I_q(p) = \frac{V_q(p)}{Z_C}$$

the simplest forms obtainable with $\rho_S = \rho_T = 0$.

Starting with a fourpole line composed of like but unsymmetrical fourpoles, we can make it symmetrical by turning every second fourpole around, so that each pair constitutes an over-all new symmetrical fourpole. We can express the new pair-parameters (\mathfrak{A}' , \mathfrak{B}' , \mathfrak{C}' , \mathfrak{D}') in terms of the original parameters (\mathfrak{A} , \mathfrak{B} , \mathfrak{C} , \mathfrak{D}) by (2.33) and find at once

$$\cosh \Gamma' = 2\mathfrak{A}^2 - 1, \quad Z_{C'} = \sqrt{\frac{\mathfrak{A}\mathfrak{B}}{\mathfrak{C}\mathfrak{D}}}$$

This supposes that we have an even number N of fourpoles to begin with. If the number N is odd, complications set in for which we might consult Guillemin,^{A8} Vol. II, pp. 167–173.

We shall restrict ourselves from here on to symmetrical fourpoles only. In addition to perfect matching, other important types of terminations for the study of transient behavior are open-circuit and short-circuit. For *open circuit* at the far end we must assume $Z_T = \infty$ which makes $\rho_T = +1$ in (15), characterizing it as a *voltage* reflection coefficient. This permits contraction of the exponentials in the numerators of (16) and (17) into hyperbolic functions. If we also

replace the exponentials in the denominator, we obtain

$$\begin{aligned} V_q(p) &= V_i(p) \frac{\cosh (N-q) \Gamma}{\cosh N \Gamma + (Z_S/Z_C) \sinh N \Gamma} \\ I_q(p) &= \frac{V_i(p)}{Z_C} \frac{\sinh (N-q) \Gamma}{\cosh N \Gamma + (Z_S/Z_C) \sinh N \Gamma} \end{aligned} \quad (22)$$

We observe here that special cases arise with various choices of Z_S ; the simplest form is when $Z_S = 0$ which we shall often find convenient.

For *short circuit* at the far end we must assume $Z_T = 0$ which makes $\rho_T = -1$ in (15). This again permits contraction of the numerators into hyperbolic functions, leading now to the forms

$$\begin{aligned} V_q(p) &= V_i(p) \frac{\sinh (N-q) \Gamma}{(Z_S/Z_C) \cosh N \Gamma + \sinh N \Gamma} \\ I_q(p) &= \frac{V_i(p)}{Z_C} \frac{\cosh (N-q) \Gamma}{(Z_S/Z_C) \cosh N \Gamma + \sinh N \Gamma} \end{aligned} \quad (23)$$

The results in (22) and (23) appear amenable to the use of expansion theorems if we can assure proper expansion into partial fractions. Physically, we need only to remember that we are still dealing with finite lumped-parameter networks to gain confidence. Mathematically, the trigonometric and hyperbolic functions belong to the group of *meromorphic functions** with only distinct (first-order) zeros and poles in the finite complex plane, admitting of infinite product representations; thus

$$\begin{aligned} \sin p &= -j \sinh (jp) = p \left[1 - \left(\frac{p}{\pi} \right)^2 \right] \left[1 - \left(\frac{p}{2\pi} \right)^2 \right] \\ &\quad \left[1 - \left(\frac{p}{3\pi} \right)^2 \right] \cdots \left[1 - \left(\frac{p}{\nu\pi} \right)^2 \right] \cdots \\ \cos p &= \cosh (jp) = \left[1 - \left(\frac{2p}{\pi} \right)^2 \right] \left[1 - \left(\frac{2p}{3\pi} \right)^2 \right] \\ &\quad \left[1 - \left(\frac{2p}{5\pi} \right)^2 \right] \cdots \left[1 - \left(\frac{2p}{(2\nu-1)\pi} \right)^2 \right] \cdots \end{aligned} \quad (24)$$

We can employ these root factors for expansions into linear partial fractions, even though their number may be infinite, in accordance

* L. Bieberbach, *Lehrbuch der Funktionentheorie*, chapter XIII, G. B. Teubner, Leipzig, 1934; reprinted by Chelsea Publishing Company, New York, 1945. K. Knopp, *Theory of Functions*, Dover Publications, New York, 1945.

with the Mittag-Leffler theorem demonstrated in function theory, and apply the conventional form of the expansion theorem to obtain the inverse Laplace transform; see Churchill,^{D3} pp. 44–51, and Doetsch,^{D5} pp. 139–143. The only requirement is that no poles of the function of p , which is a Laplace transform and of which we need to find the inverse Laplace transform, occur to the right of the path of integration defining the inverse Laplace transform (section 1.4). The proof of the validity of the expansion theorem rests primarily upon the fact that the poles to the left of the path of integration are distinct and remain equidistant along a radius from the origin even as we approach infinity, so that they are “countable” up to any large number desired; This assures convergence of the inverse Laplace integral for any finitely large semicircle closing the path over the left half-plane as in Fig. 1.11.

3.3 The Finite R-C Line

We consider as the individual fourpole a symmetrical T-section shown in Fig. 3.2a. In the terminology of the general T-section, Fig. 2.4a, we have

$$Z_1 = Z_2 = \frac{R}{2}, \quad Y = pC$$

so that the general fourpole parameters (2.26) follow as

$$\begin{bmatrix} \alpha & \beta \\ \epsilon & \alpha \end{bmatrix}_T = \begin{bmatrix} 1 + \frac{RpC}{2} & R \left(1 + \frac{RpC}{4} \right) \\ pC & 1 + \frac{RCp}{2} \end{bmatrix} \quad (25)$$

As a symmetrical fourpole line, the propagation function Γ is computed by (20) as

$$\cosh \Gamma = 1 + \frac{RpC}{2} = 1 + pT \quad (26)$$

if we introduce the time constant of the half-section $T = \frac{1}{2}RC$. The characteristic impedance becomes with (21) and (25)

$$Z_c = \sqrt{\frac{R}{pC} \left(1 + \frac{RCp}{4} \right)} = \frac{1}{pC} \sqrt{2pT + (pT)^2} = \frac{\sinh \Gamma}{pC} \quad (27)$$

The complete Laplace transform solution for the *short-circuited line* with $Z_s = R'$ and with the current source indicated in Fig. 3.2b, becomes then

$$V_q(p) = R' I_i(p) \frac{\sinh(N - q)\Gamma}{(R'pC)/(\sinh \Gamma) \cosh N\Gamma + \sinh N\Gamma} \quad (28)$$

$$I_q(p) = R' I_i(p) \frac{\cosh(N - q)\Gamma}{R' \cosh N\Gamma + (\sinh \Gamma)/(pC) \sinh N\Gamma}$$

It is obviously advantageous to presume an ideal current source so that we can let $R' \rightarrow \infty$. To apply this to (28), we first divide numerator

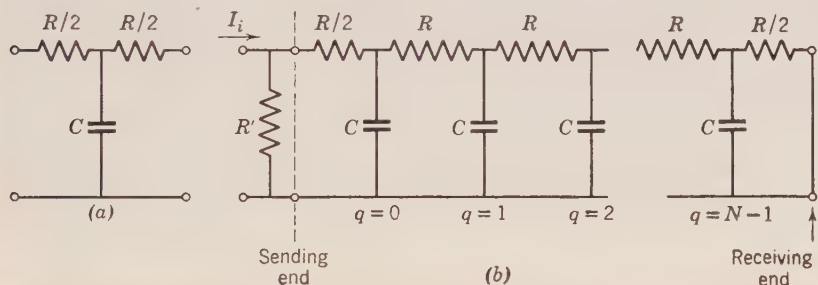


Fig. 3.2. Short-circuited R-C line: (a) individual fourpole, (b) line with current source.

and denominator by R' and then put $R' = \infty$, which leads to the more manageable forms

$$V_q(p) = I_i(p) \frac{\sinh \Gamma \sinh(N - q)\Gamma}{pC \cosh N\Gamma} \quad (29)$$

$$I_q(p) = I_i(p) \frac{\cosh(N - q)\Gamma}{\cosh N\Gamma}$$

Let us first find the current response of the line to an applied step current $I_m 1$ with the Laplace transform

$$I_i(p) = \mathcal{L} I_m 1 = \frac{I_m}{p}$$

The line current transform is then

$$I_q(p) = I_m \frac{\cosh N\Gamma \cosh q\Gamma - \sinh N\Gamma \sinh q\Gamma}{p \cosh N\Gamma} \quad (30)$$

As outlined in section 3.2, we can employ the conventional expansion theorem or the *sum of residues*, if we prefer this term. The root values of the denominator are $p = 0$, and from the second factor if we recall (24),

$$\cosh N\Gamma = 0, \quad \Gamma_\nu = -j \frac{2\nu - 1}{2N} \pi \quad \nu = 1, 2, 3 \dots$$

or

$$p_\nu = -\frac{1}{T} \left(1 - \cos \frac{2\nu - 1}{2N} \pi \right) \quad \nu = 1, 2, 3 \cdots N \quad (31)$$

Though $\cosh N\Gamma$ has an infinity of zeros, p_ν can take only N different values which make $\cosh N\Gamma$ zero; the integer values $\nu > N$ reproduce the same values of p_ν over and over again. For example, $\nu = N$ gives $\cos [\pi - (\pi/2N)]$, $\nu = N + 1$ gives $\cos [\pi + (\pi/2N)] \equiv \cos [\pi - (\pi/2N)]$, etc. Indeed, we should feel quite gratified because the entire line as a network has N meshes (since $R' \rightarrow \infty$) and as single energy system should therefore exhibit exactly N transient modes!

We obtain the time solution by means of the expansion theorem for the indicial admittance from (57) or Table 1.4, line 12. At $p = 0$ we have the simple residue I_m . For the roots p_ν we need the derivative of the denominator function

$$\frac{d}{dp} \cosh N\Gamma = N \sinh N\Gamma \frac{d\Gamma}{dp}$$

and we obtain the last derivative from (26), differentiating both sides with respect to their variables,

$$\sinh \Gamma \, d\Gamma = T \, dp, \quad \frac{d\Gamma}{dp} = \frac{T}{\sinh \Gamma}$$

If we now form from (30)

$$\frac{\text{Num}(p)}{p(d/dp)(\cosh N\Gamma)} = \frac{\cosh N\Gamma \cosh q\Gamma - \sinh N\Gamma \sinh q\Gamma}{pNT \sinh N\Gamma} \sinh \Gamma$$

and introduce the root values p_ν , we see that the first part of the numerator vanishes because $\cosh N\Gamma_\nu = 0$, and $\sinh N\Gamma$ cancels in the second part. Thus, we have as the final solution

$$i_q(t) = I_m \left\{ 1 - \sum_{\nu=1}^N \frac{\sin \left(\frac{2\nu - 1}{N} \frac{\pi}{2} \right) \sin \left(\frac{2\nu - 1}{N} q \frac{\pi}{2} \right)}{N \left[1 - \cos \left(\frac{2\nu - 1}{N} \frac{\pi}{2} \right) \right]} e^{-\frac{t}{T_\nu}} \right\} \quad (32)$$

The first term represents the steady-state current which must be the same as the impressed current value. The transient part is composed of N exponential modes, each with a different finite time constant

$$T_\nu = \frac{T}{1 - \cos \left(\frac{2\nu - 1}{N} \frac{\pi}{2} \right)} \quad \nu = 1, 2, \cdots N$$

the last one, $\nu = N$, being the shortest one, namely $T = RC/2$. We can verify that at $t = 0^+$ the entering current in the first section $q = 0$ is $i_0 = I_m$ because the sum vanishes on account of the factor $\sin \{[(2\nu - 1)/N]q(\pi/2)\}$. However, the leaving current which is also the entering current in section $q = 1$ is zero, because the terms in the sum have the form

$$\frac{\sin^2 X_\nu}{N(1 - \cos X_\nu)} = \frac{1}{N} (1 + \cos X_\nu)$$

The cosine terms are symmetrically arranged about $\pi/2$ and so cancel pairwise, leaving as a total value for the sum just unity, so that $i_1 = 0$.

For $t > 0$, $i_0(t) = I_m$ remains true, whereas all the other section currents $i_q(t)$ build up exponentially with N time constants. The expression becomes simplest for $q = N$, when $\sin [(2\nu - 1)\pi/2] = (-1)^{\nu-1}$, and

$$\frac{\sin X}{1 - \cos X} = \frac{2 \sin (X/2) \cos (X/2)}{2 \sin^2 (X/2)}$$

so that

$$i_N(t) = I_m \left[1 + \frac{1}{N} \sum_{\nu=1}^N (-1)^\nu \cot \left(\frac{2\nu - 1}{2N} \frac{\pi}{2} \right) \sin \left(\frac{2\nu - 1}{N} q \frac{\pi}{2} \right) e^{-\frac{t}{T_\nu}} \right]$$

For the voltage transform of the short-circuited line we have from (29), if we introduce again $I_i(p) = I_m/p$

$$V_q(p) = I_m \frac{\sinh \Gamma \sinh (N - q)\Gamma}{ppC \cosh N\Gamma} \quad (33)$$

The denominator appears to have a double root at $p = 0$; the numerator might at first glance appear irrational, because we have from (26)

$$\sinh \Gamma = \sqrt{pT} \sqrt{2 + pT} = [\cosh^2 \Gamma - 1]^{1/2} \quad (34)$$

However, even though $\sinh \Gamma$ by itself as an irrational function would have to be treated with due respect for its branch points, it is joined in the numerator of (33) by $\sinh (N - q)\Gamma$ and this product is a rational function of p exactly as $\cosh N\Gamma$ is because of (26). This *must* be so because we are dealing with a finite lumped-parameter network! Specifically, it can be shown* that

* See H. S. Carslaw, *Plane Trigonometry*, 3rd edition, Macmillan, London, 1930, p. 119; Carslaw-Jaeger,^{D2} p. 43.

$$\begin{aligned}
 \cosh N\Gamma &= 2^{N-1} \left(\cosh \Gamma - \cos \frac{\pi}{2N} \right) \left(\cosh \Gamma - \cos \frac{3\pi}{2N} \right) \cdots \\
 &\quad \left(\cosh \Gamma - \cos \frac{(2N-1)\pi}{2N} \right) \\
 \sinh N\Gamma &= 2^{N-1} \sinh \Gamma \left(\cosh \Gamma - \cos \frac{\pi}{N} \right) \left(\cosh \Gamma - \cos \frac{2\pi}{N} \right) \cdots \\
 &\quad \left(\cosh \Gamma - \cos \frac{(N-1)\pi}{N} \right)
 \end{aligned} \tag{35}$$

which puts in evidence the roots (31) for the function $\cosh N\Gamma$ and clearly separates a factor $\sinh \Gamma$ from $\sinh N\Gamma$, or, of course, similarly from $\sinh (N - q)\Gamma$, so that the numerator in (33) has effectively a factor $\sinh^2 \Gamma$ which is definitely a rational function in p from (34) and has a zero of first order at $p = 0$. Thus (33) has exactly the same total set of root values as the current transform (30) and can be treated in the same manner.

For the steady-state value we need to find

$$\lim_{p \rightarrow 0} pV_q(p) = I_m \lim_{p \rightarrow 0} \frac{\sinh \Gamma \sinh (N - q)\Gamma}{pC \cosh N\Gamma}$$

Application of L'Hopital's rule does not succeed, because of the product form in numerator and denominator which reproduces the form 0/0. Because $p = 0$ coincides with $\Gamma = 0$, we can expand the hyperbolic functions near the origin into their Taylor series,

$$\sinh \Gamma \cong \Gamma, \quad \sinh (N - q)\Gamma \cong (N - q)\Gamma, \quad \cosh N\Gamma \cong 1$$

and with (26),

$$pC = \frac{2}{R} (\cosh \Gamma - 1) \approx \frac{2}{R} \frac{\Gamma^2}{2}$$

and thus obtain the steady-state solution

$$(v_q)_{ss} = I_m R(N - q) \tag{36}$$

This is indeed the expected result, showing the linear voltage decrease along the fourpole line, the voltage becoming zero for $q = N$.

The complete solution is obtained in the same manner as that for the current and can be contracted into

$$v_q(t) = I_m \left[R(N - q) - \frac{R}{2} \frac{1}{N} \sum_{\nu=1}^N \cot^2 \left(\frac{2\nu-1}{2N} \frac{\pi}{2} \right) \cos \left(\frac{2\nu-1}{N} q \frac{\pi}{2} \right) e^{-\frac{t}{T_\nu}} \right]$$

If we had chosen $R' = R/2$ as the input shunt resistance, then the voltage transform in (29) would retain in the denominator as first factor

$$\frac{RpC}{2 \sinh \Gamma} = \frac{\cosh \Gamma - 1}{\sinh \Gamma} = \frac{\sinh (\Gamma/2)}{\cosh (\Gamma/2)}$$

where we have used (26) to replace $RpC/2$. Thus

$$V_q(p) = \frac{R}{2} I_i(p) \frac{\cosh (\Gamma/2) \sinh (N - q)\Gamma}{\sinh (N + \frac{1}{2})\Gamma} \quad (37)$$

Taking again for the applied current the step function $I_m 1$ with transform $I_i(p) = I_m/p$, we find for the steady-state voltage distribution

$$(v_q)_{ss} = \frac{\frac{R}{2} I_m}{N + \frac{1}{2}} (N - q) = RI_m \frac{N - q}{2N + 1} \quad (38)$$

where the first form expresses the physical fact that $(R/2)I_m$ is the equivalent applied step voltage if we convert to a voltage source. It is interesting to compare this result with (36), where R' was assumed infinitely large. The complete solution follows by the same method used above.

The RC ladder network was used rather early to simulate the electrical response characteristics of a long cable in which the inductance and leakance were disregarded; it is therefore frequently referred to as *artificial cable*.* In most treatments, a step voltage is applied at the input terminals. Let us therefore consider the application of a step voltage from an ideal source, so that $R' = 0$. From (29) we now obtain, replacing $R'I_i(p)$ by $V_i(p) = V_m/p$ as the voltage source transform

$$\begin{aligned} V_q(p) &= V_m \frac{\sinh (N - q)\Gamma}{p \sinh N\Gamma} \\ I_q(p) &= CV_m \frac{\cosh (N - q)\Gamma}{\sinh \Gamma \sinh N\Gamma} \end{aligned} \quad (39)$$

We observe the difference in the denominators here and in (29) leading to a different set of root values. As demonstrated for (33), the product of the two \sinh functions in the denominator makes the current transform a rational function in p , and, in fact, with (35) and (34) the root values are

* A. E. Kennelly, *Artificial Electric Lines*, McGraw-Hill, New York, 1917.

$$p = 0, \quad p = -\frac{2}{T}; \quad p_\nu = -\frac{1}{T} \left(1 - \cos \frac{\nu\pi}{N} \right) \\ \nu = 1 \text{ to } (N - 1) \quad (40)$$

The first two roots come from $\sinh^2 \Gamma = 0$. We have here actually $N + 1$ root values because there are $N + 1$ meshes, the root from the applied voltage having cancelled out. In applying the expansion theorem we can evaluate the total response as if the first two roots in (40) were part of the series ν because they can be identified with $\nu = 0$ and $\nu = N$, respectively. We are therefore using here the expansion theorem in the form (1.55), so that we need not split off the root $p = 0$

$$i_q(t) = CV_m \sum_{\nu=0}^N \left(\frac{\cosh (N - q)\Gamma}{(d/dp)(\sinh \Gamma \sinh N\Gamma)} e^{pt} \right)_{p=p_\nu}$$

The denominator becomes

$$(\cosh \Gamma \sinh N\Gamma + N \sinh \Gamma \cosh N\Gamma) \frac{d\Gamma}{dp}$$

and from (26) again

$$\frac{d\Gamma}{dp} = \frac{CR}{2 \sinh \Gamma}$$

With the root values from (40) we then find

$$i_q(t) = \frac{V_m}{NR} \left[1 + (-1)^q e^{-\frac{t}{T_N}} + 2 \sum_{\nu=1}^{N-1} \cos \left(\frac{\nu q}{N} \pi \right) e^{-\frac{t}{T_\nu}} \right]$$

where the time constants follow from (40)

$$T_\nu = \frac{T}{1 - \cos (\nu\pi/N)} \quad \nu = 1, 2, \dots N$$

We can check that for $t = 0^+$ the initial current distribution satisfies the physical conditions. We have in general

$$i_q(0^+) = \frac{V_m}{NR} \left[1 + (-1)^q + 2 \sum_{\nu=1}^{N-1} \cos \left(\frac{\nu q}{N} \pi \right) \right]$$

For $q = 0$ the summation leads to $2(N - 1)$, so that $i_0(0^+) = (2V_m)/R$ as it should, since the first shunt condenser represents at the first instant a short circuit. For $q = 1$, the arguments of the cosine terms

are symmetrical about $\pi/2$ and cancel exactly in pairs even if N is even, because then the center value $\nu = N/2$ gives zero value; thus the current $i_1(0^+) = 0$ and this holds, of course, for all others, so that $i_q(0^+) = 0, q \geq 1$.

The selection of voltage and current sources and of terminations could be further varied, but the method of solution will remain essentially the same, primarily the use of the expansion theorems. Thus, the open-circuited fourpole line with step voltage applied is treated in Jaeger,^{D11} p. 61.

3.4 The Indicial Admittance of Finite Ladder Wave Filters

The cascade connection of either symmetrical T-sections as in Fig. 2.4e with $Z_1 = Z_2 = Z/2$ or of symmetrical Π -sections as in Fig. 2.4d with $Y_1 = Y_2 = Y/2$ results in a uniform ladder structure as shown in Fig. 3.3a. From (2.25) and (2.23), respectively, we can take the general parameters with these special values of series and shunt arms, so that

	T-Section	Π -Section	
$\alpha = \mathfrak{D} =$	$1 + \frac{1}{2}YZ$	$1 + \frac{1}{2}YZ$	
$\mathfrak{B} =$	$Z(1 + \frac{1}{4}YZ)$	Z	(41)
$\mathfrak{C} =$	Y	$Y(1 + \frac{1}{4}YZ)$	

$Z_c = \sqrt{\frac{\mathfrak{B}}{\mathfrak{C}}} =$	$\left(\frac{Z}{Y}\right)^{\frac{1}{2}} \left(1 + \frac{1}{4}YZ\right)^{+\frac{1}{2}}$	$\left(\frac{Z}{Y}\right)^{\frac{1}{2}} \left(1 + \frac{1}{4}YZ\right)^{-\frac{1}{2}}$	
$\cosh \Gamma = \alpha =$	$1 + \frac{1}{2}YZ$	$1 + \frac{1}{2}YZ$	(42)
$\sinh \frac{\Gamma}{2} =$	$\frac{1}{2} \sqrt{YZ}$	$\frac{1}{2} \sqrt{YZ}$	

This tabulation shows the same propagation constant for the two individual fourpole sections but different characteristic impedances. The former is quite natural because the structure of Fig. 3.3a can be interpreted as made up of either kind of sections. The latter is indicative of the difference in entrance and exit sections; this difference is perhaps most obvious if we look at Figs. 3.3b and c, where the appropriate source function is directly applied at the terminals of the entrance section $q = 0$. The two cases are called *mid-series* and *mid-shunt* terminations, respectively, and more frequently than not one chooses the indicated kind of source function because it lends itself particularly well to incorporation into the network.

Let us first choose a *low-pass filter* with $Z = pL$, $Y = pC$, composed of N complete T-sections and *short-circuited* at the far end. At the sending end we assume the termination of Fig. 3.3b, namely a step

voltage $V_m I$ applied at the mid-series of the section $q = 0$. We obtain the propagation constant from (42)

$$\cosh \Gamma = 1 + \frac{1}{2} p^2 LC = 1 + \left(\frac{p}{\Omega_0} \right)^2 \quad (43)$$

where $\Omega_0 = \sqrt{2/(LC)}$ is the undamped natural angular frequency of a series circuit of $L/2$ and C , i.e., one half T-section. We can then identify

$$\sinh^2 \Gamma = \cosh^2 \Gamma - 1 = \left(\frac{p}{\Omega_0} \right)^2 \left[2 + \left(\frac{p}{\Omega_0} \right)^2 \right] \quad (44)$$

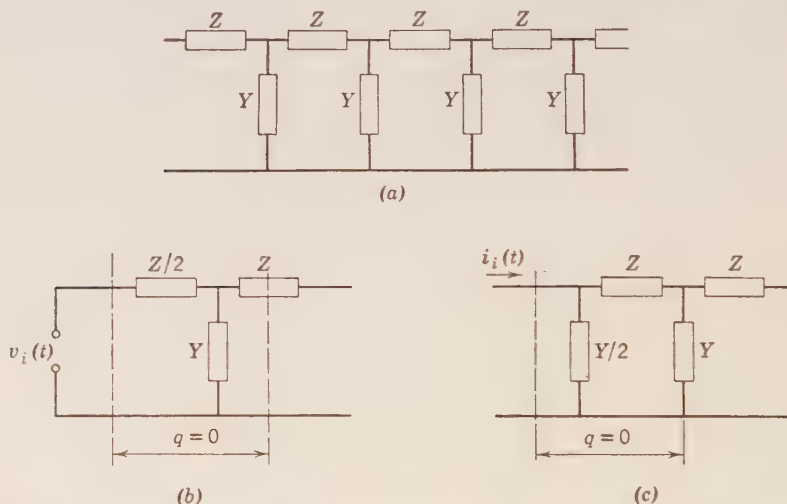


Fig. 3.3. The finite wave filter: (a) section of general symmetrical ladder network, (b) mid-series termination, (c) mid-shunt termination.

and have for the characteristic impedance, similar to (27)

$$Z_c = \sqrt{\frac{L}{C}} \left[1 + \frac{1}{2} \left(\frac{p}{\Omega_0} \right)^2 \right]^{1/2} = \frac{1}{pC} \sinh \Gamma \quad (45)$$

Referring back to (23), we set $Z_s = 0$ and with (45) obtain for the current transform

$$I_q(p) = CV_m \frac{\cosh (N - q)\Gamma}{\sinh \Gamma \sinh N\Gamma} \quad (46)$$

which is exactly the same expression as (39) for the artificial cable. We can therefore take the same root values in terms of Γ , but the translation into corresponding root values of p is quite different. From

(35) we see that we can separate a factor $\sinh \Gamma$ from $\sinh N\Gamma$, so that $\sinh^2 \Gamma = 0$ delivers the roots indicated by (44),

$$p = 0 \text{ (double root),} \quad p = \pm j\Omega_0 \sqrt{2} \quad (47)$$

The other factors in (35) give the series of root values with (43)

$$\cosh \Gamma_\nu = \cos \frac{\nu\pi}{N}, \quad p_\nu = \pm j\Omega_0 \left(1 - \cos \frac{\nu\pi}{N}\right)^{1/2} \\ \nu = 1, 2 \cdots (N-1) \quad (47a)$$

It is obvious from (47) and (47a) that the transient response will consist of sustained oscillations with angular frequencies lying between zero and $\Omega_0 \sqrt{2} = \Omega_c$ so that we confirm the nature of a low-pass filter with cutoff angular frequency Ω_c .

We might now stop to compare these results with those obtained in section 2.3, particularly from (2.53) applied to several T-sections in cascade. In this comparison we must observe that we defined the series branch in section 2.3 as $2L$, whereas here we have defined it as L . The meaning of Ω_c is therefore relatively the *same* in both cases even though it appears different by $\sqrt{2}$. If we now choose $N = 1$ in (46) we have the denominator $\sinh^2 \Gamma$ and the roots (47); but this is the same as for the single section in (2.53). If we take $N = 2$, we get (47) as well as a root pair $p = \pm j\Omega_0 = \pm j\Omega_c/\sqrt{2}$ from (47a) since $\cos \pi/2 = 0$; but for two sections in (2.53) we find the same additional natural frequency! We can go on and verify that the brief table of natural frequencies given near the end of section 2.3 is nothing else than the explicit enumeration of the natural frequencies defined by (47) and (47a).

The further evaluation of the actual response as the inverse Laplace transform of (46) follows exactly the applications of the expansion theorem in the previous examples. Inasmuch as $p = 0$ is a pole of second order in (46), we need to calculate the residue there separately, e.g., by application of Cauchy's relation as shown in (1.69). However, if we wanted to apply that relation, namely find

$$\left[\frac{d}{dp} \left(\frac{p^2 \cosh (N-q)\Gamma}{\sinh \Gamma \sinh N\Gamma} e^{pt} \right) \right]_{p=0}$$

we might be discouraged by the unwieldy differentiation, necessitating in addition application of L'Hopital's rule. A much simpler approach is the direct demonstration of the residue as the coefficient of the negative power of first order in a Laurent expansion about the pole. We can proceed as in preparation for (36) and introduce the expansions

$$\cosh (N - q)\Gamma = 1 + \frac{(N - q)^2}{2} \Gamma^2 + \dots ;$$

$$e^{pt} = 1 + pt + \frac{(pt)^2}{2} + \dots$$

$$\sinh \Gamma = \Gamma \left(1 + \frac{\Gamma^2}{6} + \dots \right);$$

$$\sinh N\Gamma = N\Gamma \left(1 + \frac{(N\Gamma)^2}{6} + \dots \right)$$

directly into the integrand of the inverse Laplace transform of (46), which is

$$i_q(t) = \mathfrak{L}^{-1} I_q(p) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} CV_m \left(\frac{\cosh (N - q)\Gamma}{\sinh \Gamma \sinh N\Gamma} e^{pt} \right) dp$$

The terms in the large parentheses give then the following expansions in numerator and denominator

$$\frac{1 + pt + \frac{(N - q)^2}{2} \Gamma^2 + \frac{(pt)^2}{2} + \dots}{N\Gamma^2 [1 + \frac{1}{6}(1 + N^2)\Gamma^2 + \dots]}$$

The relation between p and Γ near the origin $p = 0$ is again furnished by (44), where we can expand $\sinh^2 \Gamma$ as above and find as first-order approximation $\Gamma^2 = p^2/\Omega_0^2$, so that the long-hand division of the above expansion gives the first terms of the Laurent series

$$\frac{\Omega_0^2}{N} \left[\frac{1}{p^2} + \frac{t}{p} + \left(\frac{(N - q)^2}{2\Omega_0^2} - \frac{1}{6} \frac{N^2 + 1}{\Omega_0^2} + \frac{t^2}{2} \right) + \dots \right]$$

From this series we read as residue the coefficient to p^{-1} , namely

$$\frac{\Omega_0^2}{N} t = \frac{1}{NLC} t$$

or as the contribution of the double pole to the current

$$i'(t) = \frac{V_m}{NL} t \quad (48)$$

We could have established this as the “steady-state” solution of the problem because the fourpole line responds to a constant voltage as a single series inductance of value (NL) .

All the other root values occur as conjugate imaginary pairs. We find their contributions by the direct application of the expansion theorem in the form (1.55) since we have already taken care of the pole

$p = 0$. We therefore need

$$\frac{d}{dp} (\sinh \Gamma \sinh N\Gamma) = \frac{d\Gamma}{dp} (\cosh \Gamma \sinh N\Gamma + N \sinh \Gamma \cosh N\Gamma) \quad (49)$$

With (43) we get readily

$$\sinh \Gamma \, d\Gamma = \frac{2p}{\Omega_0^2} dp$$

so that we can obtain the derivative in (49). Since we may consider the simple root pair $\pm j\Omega_0 \sqrt{2}$ as part of the series in (47a), namely $\nu = N$, we can take for all these roots from (47) and (47a)

$$\cosh \Gamma_\nu = \cos \frac{\nu\pi}{N}, \quad \sinh \Gamma_\nu = -j \sin \frac{\nu\pi}{N}$$

$$\cosh N\Gamma_\nu = (-1)^\nu, \quad \sinh N\Gamma_\nu = 0$$

Expanding the numerator of (46) and introducing (49) for the denominator, we find that the conjugate pairs of the roots lead to these conjugate pairs of coefficients from (46)

$$CV_m \frac{\Omega_0^2}{2} \cos \left(\frac{q\nu}{N} \pi \right) \left(\frac{e^{p_\nu t}}{p_\nu} + \frac{e^{p_\nu^* t}}{p_\nu^*} \right)$$

The total sum becomes then as the transient part of the current

$$i_q''(t) = \frac{2V_m}{NL} \sum_{\nu=1}^N \cos \left(\frac{\nu q}{N} \pi \right) \frac{\sin \Omega_\nu t}{\Omega_\nu} \quad (50)$$

with

$$\Omega_\nu = \Omega_0 \sqrt{1 - \cos \frac{\nu\pi}{N}}$$

The transient part is composed of exactly N sustained natural oscillations with the spectrum of the angular frequencies defined by $0 \leq \Omega_\nu \leq \Omega_c$, and becoming denser as N increases.

Let us next consider a *band-pass filter* as shown in Fig. 3.4 with

$$\begin{aligned} Z &= pL_1 + \frac{1}{pC_1} = \frac{L_1}{p} (p^2 + \Omega^2) \\ Y &= pC_2 + \frac{1}{pL_2} = \frac{C_2}{p} (p^2 + \Omega^2) \\ YZ &= \frac{L_1 C_2}{p^2} (p^2 + \Omega^2)^2 \end{aligned} \quad (51)$$

where we have chosen

$$\Omega^2 = (L_1 C_1)^{-1} = (L_2 C_2)^{-1}$$

At the sending end we assume now the termination of Fig. 3.3c, namely a step current $I_m 1$ applied at *mid-shunt* of section $q = 0$. The

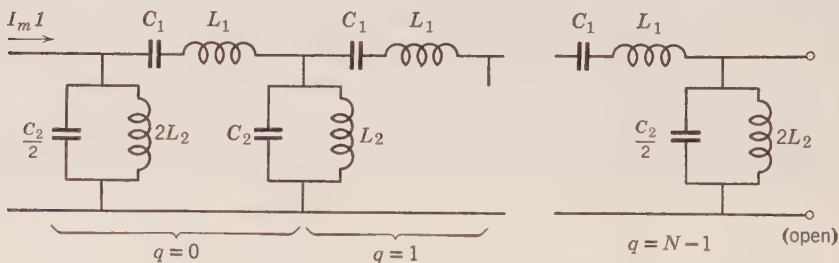


Fig. 3.4. Finite band-pass filter open-circuited at far end; $L_1 C_1 = L_2 C_2 = \Omega^{-2}$.

far end we take here as *open-circuited*, so that the general form of the voltage is from (22) with $Z_s \rightarrow \infty$ for the ideal current source

$$V_q(p) = I_i(p) Z_c \frac{\cosh (N - q) \Gamma}{\sinh N \Gamma} \quad (52)$$

From (42) we obtain the propagation constant

$$\cosh \Gamma = 1 + \frac{YZ}{2} = 1 + \frac{1}{2} \frac{L_1 C_2}{p^2} (p^2 + \Omega^2)^2 \quad (53)$$

We can identify by (42) also

$$\sinh \frac{\Gamma}{2} = \pm \left(\frac{1}{2} (\cosh \Gamma - 1) \right)^{1/2} = \pm \frac{1}{2} \frac{\sqrt{L_1 C_2}}{p} (p^2 + \Omega^2)$$

for later use in the root values. The characteristic impedance is again expressed advantageously by

$$Z_c = \frac{Z}{\sinh \Gamma} = L_1 \frac{p^2 + \Omega^2}{p \sinh \Gamma}$$

Thus, the voltage transform (52) takes the form

$$V_q(p) = I_m L_1 \frac{p^2 + \Omega^2}{p^2} \frac{\cosh (N - q) \Gamma}{\sinh \Gamma \sinh N \Gamma}$$

very similar to the current transform (46) for the low-pass filter. We can therefore proceed as in (44),

$$\sinh^2 \Gamma = \cosh^2 \Gamma - 1 = \frac{L_1 C_2}{2p^2} (p^2 + \Omega^2)^2 \left(2 + \frac{L_1 C_2}{2p^2} (p^2 + \Omega^2)^2 \right)$$

Clearly, if we use this in $V_q(p)$, the factor $(p^2 + \Omega^2)/p^2$ cancels and we have left $(p^2 + \Omega^2)$ and the bracketed factor which give the root values, respectively,

$$p = \pm j\Omega, \quad p = \pm j \frac{1}{\sqrt{L_1 C_2}} \pm j \sqrt{\Omega^2 + \frac{1}{L_1 C_2}} = \pm j \begin{cases} \Omega_c' > 0 \\ \Omega_c'' > 0 \end{cases} \quad (54)$$

The remainder of $\sinh N\Gamma$ gives from (35) the same result as (47a), only that we must now use $\cosh \Gamma$ from (53), or better, the simpler expression for $\sinh (\Gamma/2)$ which also relates $\cosh \Gamma$ and p ; thus

$$\begin{aligned} \cosh \Gamma_\nu &= \cos \frac{\nu\pi}{N}, & \sinh \frac{\Gamma_\nu}{2} &= \pm \left[\frac{1}{2} \left(\cos \frac{\nu\pi}{N} - 1 \right) \right]^{1/2} \\ & & &= \pm \frac{1}{2} \frac{\sqrt{L_1 C_2}}{p_\nu} (p_\nu^2 + \Omega^2) \\ p_\nu &= \pm j \frac{\sin \frac{\nu\pi}{2N}}{\sqrt{L_1 C_2}} \pm j \sqrt{\Omega^2 + \frac{\sin^2 \frac{\nu\pi}{2N}}{L_1 C_2}} = \pm j \begin{cases} \Omega_\nu' \\ \Omega_\nu'' \end{cases} \end{aligned} \quad (55)$$

This indicates four different root values of p for every root Γ_ν , or two pairs of conjugate imaginary roots that will result in a real physical solution, and the pairs of different values have as a geometric mean just Ω^2 , which is a characteristic of the band-pass filter as we will see in section 3.5, namely

$$p_\nu' p_\nu'' = j^2 \Omega_\nu' \Omega_\nu'' = -\Omega^2$$

The actual solution, since there is no steady-state root left, is found by the use of the expansion theorem in the form (1.55), quite similar to the transient part i'' in (50) of the last example. We have here from (53)

$$\sinh \Gamma \, d\Gamma = L_1 C_2 \frac{(p^2 - \Omega^2)(p^2 + \Omega^2)}{p^3} dp$$

and for any one conjugate pair of root values we can contract their contribution into the real form

$$\frac{2I_m}{NC_2} \cos \left(\frac{\nu\pi}{N} q \right) \frac{\Omega_\nu}{\Omega^2 + \Omega_\nu^2} \sin \Omega_\nu t$$

which holds for the Ω_ν' as well as Ω_ν'' values with $\nu = 1, 2 \cdots (N - 1)$. As a spectrum of natural frequencies, all of these Ω_ν are grouped around Ω as the geometric mean. In addition, we have the conjugate root pairs (54). The second group of root values can easily be taken into account by allowing $\nu = N$ in (55) for which the $\sin \pi/2 = 1$; again the two different values will have Ω as their geometric mean. The first conjugate pair gives the simple contribution

$$\frac{I_m}{NC_2} \cos\left(\frac{\nu\pi}{N} q\right) \frac{\sin \Omega t}{\Omega}$$

We observe that the spectrum of natural frequencies is sharply limited, with the values Ω_c' and Ω_c'' from (54) as the two extremes or cutoff frequencies, and with *all* frequencies so located about Ω that in pairs they have Ω as the geometric mean.

3.5 Propagation Characteristics of Ladder Wave Filters

If we apply a sinusoidal source function to wave filters of any kind, the transient modes of response remain the same as computed in the previous section and only the amplitudes will be different. On the other hand, the steady-state solution will contribute a new aspect which, indeed, underlies the designation "wave filter" and has made these fourpole lines of such outstanding practical importance.

Let us first illustrate the point with the example of the short-circuited low-pass filter from the previous section. If we assume zero phase angle, the applied voltage is, in complex notation, $V_m e^{j\omega t}$, where V_m is the real amplitude. The Laplace transform of the entering current in the q 'th section is thus from (46), also in complex notation

$$\bar{I}_q(p) = CV_m \frac{p \cosh (N - q)\Gamma}{(p - j\omega) \sinh \Gamma \sinh N\Gamma}$$

where we replaced V_m/p by $V_m/(p - j\omega)$. The transient part of the response is readily found to be of the same form as (50) with adjustment in the amplitude factor, namely

$$i_q''(t) = \frac{2V_m}{NL} \sum_{\nu=1}^N \cos\left(\frac{\nu\pi}{N} q\right) \frac{\omega}{\omega^2 - \Omega_\nu^2} \sin \Omega_\nu t \quad (56)$$

We see that the amplitudes of response would appear to become infinite if $\omega = \Omega_\nu$ for every "resonant frequency," i.e., applied frequency coinciding with one of the natural frequencies. In such a case, however, a pole of second order would exist and (56) could not be used.

The exact solution would lead to a time function $t \sin \Omega_0 t$, tending to infinity, as found in double-energy circuits.

The steady-state response, assuming now no coincidence of applied and natural frequencies, is obtained in complex notation as

$$\bar{v}_q'(t) = CV_m \left(\frac{p \cosh (N - q)\Gamma}{\sinh \Gamma \sinh N\Gamma} \right)_{p=j\omega} e^{j\omega t} \quad (57)$$

From (44) we see if we replace $p = j\omega$

$$\sinh \Gamma = j \frac{\omega}{\Omega_0} \sqrt{2 - \left(\frac{\omega}{\Omega_0} \right)^2} \quad (58)$$

This defines Γ as imaginary for $0 < \omega < \Omega_0 \sqrt{2}$ and as real beyond $\Omega_0 \sqrt{2}$. Designated as the propagation function in (19) and written

$$\Gamma = \alpha + j\beta$$

we call α the attenuation function and β the phase function since both are functions of frequency. Now

$$\sinh (\alpha + j\beta) = \sinh \alpha \cos \beta + j \cosh \alpha \sin \beta$$

and since (58) requires this to be a pure imaginary, it can only mean

$$\alpha = 0, \quad \sin \beta = \frac{\omega}{\Omega_0} \sqrt{2 - \left(\frac{\omega}{\Omega_0} \right)^2} \quad 0 < \omega < \Omega_0 \sqrt{2}$$

and we will call $\Omega_0 \sqrt{2} = \Omega_c = 2/\sqrt{LC}$ the *cutoff frequency* of this filter. Within the pass range of frequencies there is no attenuation and only phase shift occurs as far as the transmission of a sinusoidal wave in one direction is concerned. Obviously, at $\omega = \Omega_0$ the phase shift is $\pi/2$, and as ω increases further the phase shift approaches π at $\omega = \Omega_0 \sqrt{2} = \Omega_c$. On the other hand, for $\omega > \Omega_c$ we write

$$\sinh \Gamma = - \frac{\omega}{\Omega_0} \sqrt{\left(\frac{\omega}{\Omega_0} \right)^2 - 2} \quad \omega > \Omega_0 \sqrt{2}$$

so that we must now assume (α must remain positive in any linear passive system)

$$\sinh \alpha = \frac{\omega}{\Omega_0} \sqrt{\left(\frac{\omega}{\Omega_0} \right)^2 - 2}; \quad \cos \beta = -1, \quad \beta = \pi \quad \omega > \Omega_0 \sqrt{2}$$

We should emphasize very strongly that $\Omega_0 \sqrt{2} = \Omega_c$ is *exactly the highest natural frequency* for the transient response, and it is also the

cutoff frequency of the spectrum of unattenuated frequencies of the steady-state response!

The steady-state and the transient characteristics are most intimately related: the range of possible location of natural frequencies is also the range of unattenuated transmission of steady-state frequencies. Any periodic signal that has frequency components of its Fourier analysis outside the pass band of the filter will consequently suffer steady-state "amplitude distortion" inasmuch as the amplitudes of these harmonics will be attenuated as compared with those within the pass band. Since the phase function β varies with frequency, we see that *all* harmonic components will suffer "phase distortion," unequal phase-angle changes as they pass through the network. To retain correct phase relationships, the phase-angle change should be proportional to the frequency of the harmonics. We see that $\sin \beta$ will be linear with ω over a smaller range than the cutoff frequency Ω_c so that only the lowest harmonics would be transmitted almost without phase distortion. Although these conclusions are based only upon Γ and are therefore essentially restricted to infinite lines for which Γ is the most characteristic network function, similar arguments hold for the over-all transmission impedance or admittance, whatever the case may be. A more detailed discussion of this topic will be given in chapter 4.

Now the complete steady-state solution for the two ranges of applied frequencies can be found from (57) by introducing the specific values of Γ as determined. Thus, we have, after rationalization

$$\left. \begin{aligned} i_q'(t) &= \text{Im } \bar{i}_q'(t) = \omega C V_m \frac{-\cos(N-q)\beta}{\sin \beta \sin N\beta} \cos \omega t \\ \sin \beta &= 2 \frac{\omega}{\Omega_c} \sqrt{1 - \left(\frac{\omega}{\Omega_c}\right)^2}, \quad \sin \frac{\beta}{2} = \frac{\omega}{\Omega_c} \end{aligned} \right\} 0 < \omega < \Omega_c$$

where $\Omega_c = 2/\sqrt{LC} = \Omega_0 \sqrt{2}$. For the attenuation range we obtain

$$\left. \begin{aligned} i_q'(t) &= \omega C V_m \frac{\cosh(N-q)\alpha}{\sinh \alpha \sinh N\alpha} \cos \omega t \\ \sinh \alpha &= 2 \frac{\omega}{\Omega_c} \sqrt{\left(\frac{\omega}{\Omega_c}\right)^2 - 1}, \quad \cosh \frac{\alpha}{2} = \frac{\omega}{\Omega_c} \end{aligned} \right\} \omega > \Omega_c$$

The discussion of (58) indicates how we can take any arbitrary ladder network and study the propagation characteristics under steady-state conditions. This has been done extensively and prac-

tically all textbooks on communication networks* include chapters on the steady-state theory of wave filters.†

In many of these treatises, the so-called *image parameters* are used rather than the iterative quantities which are more naturally introduced in the discussion of transient analysis. The image impedances Z_{I1} and Z_{I2} at the input and output terminals, respectively, are defined as the terminations which result in the same network impedance looking either way at each terminal pair. At the input we have from (2.11)

$$V_1 = \mathfrak{A}V_2 + \mathfrak{B}I_2$$

$$I_1 = \mathfrak{C}V_2 + \mathfrak{D}I_2$$

and, because of the terminations

$$\frac{V_1}{I_1} = Z_{I1}, \quad \frac{V_2}{I_2} = Z_{I2}$$

In addition, reversal of the fourpole gives by (2.20)

$$V_2 = \mathfrak{D}V_1 + \mathfrak{B}I_1$$

$$I_2 = \mathfrak{C}V_1 + \mathfrak{A}I_1$$

Forming the ratios we have two equations which easily lead to the solutions

$$\begin{aligned} Z_{I1} &= \left(\frac{\mathfrak{B}\mathfrak{A}}{\mathfrak{D}\mathfrak{C}} \right)^{1/2} = (Z_{S1}Z_{O1})^{1/2} \\ Z_{I2} &= \left(\frac{\mathfrak{B}\mathfrak{D}}{\mathfrak{A}\mathfrak{C}} \right)^{1/2} = (Z_{S2}Z_{O2})^{1/2} \end{aligned} \tag{59}$$

The last forms are obvious from (2.17) and (2.18). For the symmetrical fourpole $\mathfrak{A} = \mathfrak{D}$ and we find

$$Z_{I1} = Z_{I2} = Z_C = \left(\frac{\mathfrak{B}}{\mathfrak{C}} \right)^{1/2}$$

* See particularly Guillemin,^{A8} Vol. II; H. Pender and K. McIlwain, *Electrical Engineers' Handbook, Electric Communication and Electronics*, 4th edition, pp. 6.33 to 6.62 by A. J. Grossman, Wiley, New York, 1950; and the early book by T. E. Shea.^{A17}

† The original contributions came from G. A. Campbell, "Physical Theory of the Electric Wave Filter," *Bell System Tech. J.*, **1**, 1-32 (Nov. 1922); O. J. Zobel, "Theory and Design of Uniform and Composite Wave Filters," *Ibid.*, **2**, 1-46 (1923); also, "Extensions to the Theory and Design of Electric Wave Filters," *Ibid.*, **10**, 284-341 (1931); H. W. Bode, "A General Theory of Electric Wave Filters," *J. Math. and Phys.*, **13**, 275-364 (1934).

which is the characteristic impedance defined in (21). We can also show that Z_{I1} in (59) is the characteristic impedance of the fourpole and its inverse in cascade. The combination parameters $\alpha'\beta'\epsilon'\mathfrak{D}'$ are given in (2.33) and we have

$$\frac{\beta'}{\epsilon'} = \frac{\alpha\beta}{\epsilon\mathfrak{D}}$$

Similarly, the image impedance Z_{I2} in (59) is the characteristic impedance of the inverse fourpole in cascade with the original, since we only need to interchange α and \mathfrak{D} .

The *image transfer constant* θ is defined by

$$2\theta = \ln \frac{I_1^2 Z_{I1}}{I_2^2 Z_{I2}}$$

or also by

$$\tanh \theta = \left(\frac{Z_{S1}}{Z_{O1}} \right)^{1/2} = \left(\frac{Z_{S2}}{Z_{O2}} \right)^{1/2} = \left(\frac{\beta\epsilon}{\alpha\mathfrak{D}} \right)^{1/2} \quad (60)$$

It is unaffected by reversal of the fourpole. For the symmetrical fourpole $\alpha = \mathfrak{D}$ so that from (2.19)

$$\alpha\mathfrak{D} - \beta\epsilon = \alpha^2 - \beta\epsilon = 1$$

We thus find from (20)

$$\tanh \Gamma = \frac{1}{\alpha} \sqrt{\alpha^2 - 1} = \tanh \theta$$

or identity of the image transfer constant θ and the propagation function Γ . Again, if we take the cascade combination of the fourpole with its inverse, which results in a symmetrical structure, we have

$$\cosh \Gamma = \alpha' = 2\alpha\mathfrak{D} - 1$$

and we find

$$\tanh \frac{\Gamma}{2} = \left(\frac{\beta\epsilon}{\alpha\mathfrak{D}} \right)^{1/2}$$

where the half value comes from the fact that we now have two fourpoles combined into one unit. Obviously, for steady-state design computations we will take in all of the above forms $p = j\omega$ in order to obtain the direct frequency dependence.

Of the simple basic filter types, the *constant- k filters* have gained particular importance. By definition, the two characteristic elements Y and Z of the ladder network in Fig. 3.3a must satisfy the requirement

$$Z/Y = R_s^2 \quad R_s \text{ (real)} > 0 \quad (61)$$

where R_s is a positive real constant with the significance of a resistance and in many applications called the *surge resistance*. Obviously, any ladder network in which Z and Y have the same dependence on p (or on ω in steady state) can be considered a constant- k network. The above low-pass filter was a classical example and the band-pass filter of Fig. 3.4 satisfies condition (61) if we make the same stipulation as in (57), namely that series and shunt arms have the same natural frequency. From (42) we see that with (61) satisfied, we can write

$$Z_C = R_s \left(1 + \sinh^2 \frac{\Gamma}{2} \right)^{\pm 1/2} = R_s \left(\cosh \frac{\Gamma}{2} \right)^{\pm 1}$$

where the upper sign holds for the T-, and the lower sign for the Π -sections. Since the propagation function is defined by

$$\sinh \frac{\Gamma}{2} = \frac{1}{2} \sqrt{YZ} = \frac{1}{2} R_s Y = \frac{1}{2} \frac{Z}{R_s}$$

we can characterize the constant- k filter by either series or shunt arm alone, whichever is more convenient.

Let us take as an example the band-pass filter with the parameters given in (51). The propagation characteristics are then defined by

$$\sinh \frac{\Gamma}{2} = \frac{1}{2} \left(\frac{L_1}{C_2} \right)^{1/2} \frac{C_2}{j\omega} (\Omega^2 - \omega^2) = +j \frac{\sqrt{L_1 C_2}}{2\omega} (\omega^2 - \Omega^2)$$

To be purely imaginary, we must assume for $\Gamma = \alpha + j\beta$

$$\alpha = 0, \quad \sin \frac{\beta}{2} = \frac{\sqrt{L_1 C_2}}{2\omega} (\omega^2 - \Omega^2) \quad \text{with } (-1) \leq \sin \frac{\beta}{2} \leq (+1)$$

i.e., we define here a finite range of frequencies for which unattenuated transmission is possible. If we define the two limiting frequencies as Ω_c' and Ω_c'' , we find, respectively

$$\Omega_c', \Omega_c'' = \mp \frac{1}{\sqrt{L_1 C_2}} + \sqrt{\Omega^2 + \frac{1}{L_1 C_2}} \quad (62)$$

but comparison with (54) shows at once that these *steady-state* upper and lower *cutoff frequencies* are identical with the extreme natural frequencies of the network! The *width* of the pass band is therefore

$$\Omega_c' - \Omega_c'' = \frac{2}{\sqrt{L_1 C_2}} = \frac{2}{L_1} R_s$$

if we select $\Omega_c' > \Omega_c'' > 0$ from (62).

At the band edges $\sin \beta/2 = \pm 1$, so that $\cos \beta/2 = 0$; if we retain this condition, i.e., $\beta = \pm\pi$ outside the band, then we can admit $\alpha \neq 0$ and write

$$\Gamma = \alpha + j\pi, \quad \sinh \frac{\Gamma}{2} = j \cosh \frac{\alpha}{2} \sin \left(\pm \frac{\pi}{2} \right)$$

so that

$$\cosh \frac{\alpha}{2} = \frac{\sqrt{L_1 C_2}}{2\omega} (\omega^2 - \Omega^2), \quad \beta = \pm\pi \text{ with } \cosh \frac{\alpha}{2} > 1$$

Actual plots of these relations can be found, e.g., in Guillemin,^{A8} Vol. II, p. 323.

We have dealt so far only with lossless networks, yet any practical network must have some dissipation because it is constructed of physical elements. Both the characteristic impedance Z_C as well as the propagation function Γ are analytic functions of p except for certain isolated singularities once we have restricted p to a one-valued domain. If we start with the lossless network function, say $M(j\omega)$ where M can signify any of the network characteristics, then in the neighborhood of any regular frequency ω (not too close to any of the natural frequencies Ω_n), we can employ a Taylor series expansion similar to that used in Vol. I, section 5.10. Let us write for this purpose $p = \delta + j\omega$ and define $M(p)$ as the analytic function, separating the real and imaginary part

$$M(p) = A(\delta, \omega) + jB(\delta, \omega)$$

Here we must be able to relate A and B by the Cauchy-Riemann equations for any regular point in the p -plane, so that

$$\frac{\partial A}{\partial \delta} = \frac{\partial B}{\partial \omega}, \quad \frac{\partial A}{\partial \omega} = -\frac{\partial B}{\partial \delta}$$

The Taylor expansion near the imaginary axis will then become

$$\begin{aligned} M(\delta + j\omega) &= M(j\omega) + \delta \frac{\partial M}{\partial \delta} + \cdots = A + jB + \delta \left(\frac{\partial A}{\partial \delta} + j \frac{\partial B}{\partial \delta} \right) \\ &\quad + \cdots \\ &= \left(A + \delta \frac{\partial B}{\partial \omega} + \cdots \right) + j \left(B - \delta \frac{\partial A}{\partial \omega} + \cdots \right) \end{aligned}$$

We can thus identify the adjustments in real and imaginary parts of the network characteristics as requiring *only* the knowledge of the lossless characteristic and of δ which is a measure of the losses. In

first-order approximation we can take δ as the average of all damping coefficients of coils and condensers for oscillatory circuits

$$\delta = \frac{1}{n} \sum_{\sigma} \left(\frac{R_{\sigma}}{2L_{\sigma}} + \frac{1}{2} R_{\sigma}' C_{\sigma}' \right) = \frac{\omega}{2} \left(\frac{1}{Q_L} + \frac{1}{Q_C} \right) = \frac{\omega}{Q}$$

where the second form expresses the damping in terms of the average Q 's of coils and condensers and the last form uses the equivalent fictitious Q for the total dissipation. We have finally

$$\begin{aligned} A_{\text{diss}}(\omega) &\approx A(\omega) + \frac{\omega}{Q} \frac{dB}{d\omega} \\ B_{\text{diss}}(\omega) &\approx B(\omega) - \frac{\omega}{Q} \frac{dA}{d\omega} \end{aligned} \quad (63)$$

$A(\omega)$ and $B(\omega)$ are the real and imaginary parts, respectively, of the *lossless* network characteristic; we have used ordinary derivatives because A and B are functions only of ω . The relations (63) are called Mayer's theorem* and are convenient for slightly damped systems.

3.6 The Lattice Network

In section 2.2 we discussed the basic types of fourpoles and found for the general parameters of the symmetrical lattice the following expressions

$$\begin{aligned} \alpha &= \mathfrak{D} = \frac{Q + P}{Q - P} \\ \mathfrak{B} &= \frac{2QP}{Q - P} \\ \mathfrak{C} &= \frac{2}{Q - P} \end{aligned} \quad (64)$$

From (20) and (21) we can then define as the characteristic fourpole line parameters

$$\begin{aligned} \cosh \Gamma &= \alpha = \frac{Q + P}{Q - P} \\ Z_C &= \sqrt{\frac{\mathfrak{B}}{\mathfrak{C}}} = \sqrt{QP} \end{aligned}$$

* H. F. Mayer, "Attenuation of Wave Filters in the Pass Band" [German], *Elek. Nachr. Tech.*, **10**, 335-338 (1925). See also Guillemin,^{A8} pp. 445-448; Bode,^{B2} pp. 219-220.

where P are the series impedance arms and Q the cross lattice impedance arms as also indicated in Fig. 3.5 where the voltage source is

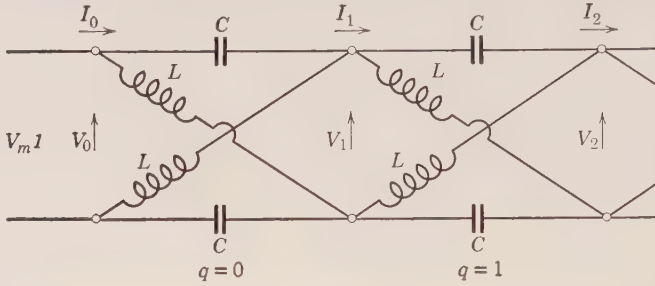


Fig. 3.5. Symmetrical lattice structure, open-circuited at far end; $LC = \Omega_0^{-2}$.

directly applied to the input terminals of section $q = 0$. Let us choose, as in the figure, $P = 1/(pC)$ and $Q = pL$; then we find

$$\cosh \Gamma = \frac{\left(\frac{p}{\Omega_0}\right)^2 + 1}{\left(\frac{p}{\Omega_0}\right)^2 - 1}, \quad \Omega_0 = \frac{1}{\sqrt{LC}} \quad (65)$$

and very simply

$$Z_c = \sqrt{\frac{L}{C}}$$

a constant value which we can again identify as surge resistance R_s as in (61).

For the open-circuited line we obtain directly from (22) with $Z_s = 0$ and with $V_i(p) = (V_m/p)$ for the Laplace transform of the step voltage

$$V_q(p) = V_m \frac{\cosh (N - q)\Gamma}{p \cosh N\Gamma} \quad (66)$$

To obtain the inverse Laplace transform we can again apply the conventional expansion theorem in any of its forms, perhaps most conveniently (1.57), because of the step voltage. The root values of the denominator can be given from (35)

$$\cosh \Gamma_\nu = \cos \frac{2\nu - 1}{N} \frac{\pi}{2} \quad \nu = 1, 2 \dots N$$

and this may be introduced directly into (65) to obtain the roots in terms of p , namely

$$p_\nu = \pm j\Omega_0 \left(\frac{1 + \cos \frac{2\nu - 1}{2N} \pi}{1 - \cos \frac{2\nu - 1}{2N} \pi} \right)^{1/2} = \pm j\Omega_\nu \quad (67)$$

The cosine values in the square root are symmetrically distributed about $\pi/2$ so that substitution of ν by $(N + 1 - \nu)$ just changes the sign of the cosine and thus the root value into its reciprocal. We therefore have

$$\Omega_\nu \Omega_{(N+1-\nu)} = \Omega_0^2$$

or a behavior reminding of the pass-band ladder network except that here the band is very wide, the network is essentially an *all-pass network* particularly useful as a corrective network for phase equalization. From (64) we observe that interchange of the elements in series and crossarms would not change this behavior.

For the steady-state response we must use the root $p = 0$ in (66). From (65) we take that

$$(\cosh \Gamma)_{p=0} = -1, \quad \Gamma_0 = j\pi, \quad \sinh \Gamma_0 = 0$$

so that from line 12 in Table 1.4

$$v_q'(t) = V_m \cos q\pi = V_m(-1)^q \quad (67a)$$

a seemingly strange result until we realize the chain of inductances presenting no voltage drop, extending crisscross like a lattice from sending end to receiving end and back. The transient response for each pair of conjugate imaginary roots (67) follows with $(d\Gamma)/(dp)$ from (65) and

$$\frac{d}{dp} (\cosh N\Gamma) = N \sinh N\Gamma \frac{d\Gamma}{dp} = -\frac{N \sinh N\Gamma}{p \sinh \Gamma} \frac{4(p/\Omega_0)^2}{[(p/\Omega_0)^2 - 1]^2}$$

as

$$+ \frac{V_m}{4N} \left[\frac{[(p/\Omega_0)^2 - 1]^2}{(p/\Omega_0)^2} \sinh \Gamma \sinh q\Gamma \right]_{p=\pm j\Omega_\nu} (e^{j\Omega_\nu t} + e^{-j\Omega_\nu t})$$

The amplitude is insensitive to the sign of p_ν because it contains only $p_\nu^2 = -\Omega_\nu^2$. The total transient response becomes thus

$$v_q''(t) = \frac{V_m}{2N} \sum_{\nu=1}^N \frac{[(\Omega_\nu/\Omega_0)^2 + 1]^2}{(\Omega_\nu/\Omega_0)^2} \sin \left(\frac{2\nu - 1}{2N} \pi \right) \sin \left(\frac{2\nu - 1}{2N} q\pi \right) \cos \Omega_\nu t$$

where we have used

$$\sinh \Gamma_\nu = -j \sin \frac{2\nu - 1}{2N} \pi, \quad \sinh q\Gamma_\nu = -j \sin \frac{2\nu - 1}{2N} q\pi$$

Further simplification can be had if we look back at (67) and introduce

$$\left(\frac{\Omega_\nu}{\Omega_0}\right)^2 = \frac{1 + \cos \mu}{1 - \cos \mu}, \quad \mu = \frac{2\nu - 1}{2N} \pi$$

This leads to

$$v_q''(t) = \frac{2V_m}{N} \sum_{\nu=1}^N \frac{\sin [(2\nu - 1)/2N] q\pi}{\sin [(2\nu - 1)/2N] \pi} \cos \Omega_\nu t \quad (68)$$

We can check the initial values. For $t = 0^+$ we have simply the sum of the amplitudes. Now, at the input of the first section $q = 0$ and therefore with the steady-state term from (67a)

$$\text{at } q = 0, \quad v_0''(0^+) = 0, \quad v_0(0^+) = V_m$$

The total voltage is the applied voltage as it should be. On the other hand, at the input of the first section $q = 1$, and (68) gives

$$\text{at } q = 1, \quad v_1''(0^+) = \frac{2V_m}{N} \sum_{\nu=1}^N 1 = 2V_m$$

Since, however, the steady-state voltage is $v_1' = -V_m$, we again have as resultant voltage the applied voltage. At $q = 2$ we can write $\sin 2\mu = 2 \sin \mu \cos \mu$, cancel $\sin \mu$, and have left

$$\sum_{\nu=1}^N \cos \frac{2\nu - 1}{2N} \pi = 0$$

so that the steady-state term maintains the applied voltage. Initially therefore, we find the applied voltage V_m all along the open-circuited lattice line.

The current in this open-circuited lattice line must start from zero, since the condensers have no connection except through inductances. From (22) we can take for the current transform in the general symmetrical fourpole line for an applied step voltage $V_m I$

$$I_q(p) = \frac{V_m}{p} \frac{1}{Z_c} \frac{\sinh (N - q)\Gamma}{\cosh N\Gamma}$$

Since $Z_C = \sqrt{L/C}$ in our case, the root values remain the same (67) as for the voltage transform. The only difference is in the numerator; because we now have $\sinh (N - q)\Gamma$, the steady-state value becomes zero since $\sinh \Gamma_0 = 0$ as shown in connection with (67a). The evaluation of the transient follows the same pattern as it does with the voltage; we just have to observe that from (65)

$$\sinh \Gamma = \frac{2(p/\Omega_0)}{(p/\Omega_0)^2 - 1}, \quad \sinh \Gamma_\nu = \mp 2j \frac{(\Omega_\nu/\Omega_0)}{(\Omega_\nu/\Omega_0)^2 + 1} = \mp j \sin \mu \quad (69)$$

Though we have used this for (68), the demonstration is necessary in order to assure the correct sign for the current terms because $\sinh \Gamma_\nu$ changes sign simultaneously with the root. Thus, the time exponentials combine now with different signs

$$(-je^{j\Omega_\nu t} + je^{-j\Omega_\nu t}) = 2 \sin \Omega_\nu t$$

and the total transient part of the current is

$$i_q''(t) = \frac{2V_m}{N\sqrt{L/C}} \sum_{\nu=1}^N \frac{\cos [(2\nu - 1)/2N]q\pi}{\sin [(2\nu - 1)/2N]\pi} \sin \Omega_\nu t$$

a solution which is quite similar to the voltage transient (68).

If we apply a sinusoidal voltage, $V_m \sin \omega t$, the transient modes remain identical with those obtained above, except that the amplitudes of the transients will change. The final expression is now

$$v_q''(t) = \frac{V_m}{2N} \sum_{\nu=1}^N \frac{[(\Omega_\nu/\Omega_0)^2 + 1]^2}{(\Omega_\nu/\Omega_0)^2 - (\omega/\Omega_0)^2} \frac{\omega}{\Omega_\nu} \sin \left(\frac{2\nu - 1}{2N} \pi \right) \sin \left(\frac{2\nu - 1}{2N} q\pi \right) \sin \Omega_\nu t$$

primarily given in order to demonstrate that the natural frequencies of the lattice line also are the "resonant frequencies" if a sinusoidal voltage is applied. Again, if $\omega = \Omega_\nu$, then we cannot use the expression just given since a pole of second order will exist requiring modifications in accordance with Table 1.1.

We might, for the sake of completeness, also check the propagation characteristics under steady-state a-c conditions. We take from (69) for $p = j\omega$

$$\sinh \Gamma = \frac{-2j(\omega/\Omega_0)}{(\omega/\Omega_0)^2 + 1}$$

This case is similar to the band-pass ladder network we have treated in the previous section. To be purely imaginary, we must assume for $\Gamma = \alpha + j\beta$ that

$$\alpha = 0, \quad \sin \beta = \frac{-2(\omega/\Omega_0)}{(\omega/\Omega_0)^2 + 1}$$

with the condition $(-1) \leq \sin \beta \leq (+1)$. This determines the range of frequencies transmitted without attenuation, and if we solve for the limiting frequencies we obtain $\omega_c = \pm \Omega_0$. Since, however, the maximum value of the right-hand side of $\sin \beta$ is unity, and for $\omega > \omega_c$, $\sin \beta$ again decreases, we must refer to Ω_0 as a critical frequency at which $\beta = \pi/2$ but which does not constitute a cutoff frequency, verifying the nature of this lattice line as an all-pass network.

Of course, with proper choices of the lattice elements we can reproduce an unlimited variety of filter characteristics. The treatment of any specific arrangement follows the simple examples shown here. Much detailed discussion of the steady-state behavior is given in Guillemin,^{A8} Vol. II, at various places and in some of the other references listed in Appendix 3A.

3.7 The Infinite Wave Filter

The concept of the infinitely long wave filter, though obviously not physically realizable, either by the requisite number of sections or in general by the required terminal impedance (which might be an irrational function of frequency), nevertheless has great attraction because in several instances one can arrive at rather simple solutions and because these solutions admit of practical conclusions of value even in the case of finite wave filters. We have observed and so stated that, e.g., a low-pass filter will add natural frequencies as the number of sections is increased, all of these frequencies lying within the region $0 < \omega < \Omega_c$ where Ω_c is the cutoff frequency (see section 3.5). If the number of sections increases to infinity one would expect that the natural frequencies would form a continuum for $0 < \omega < \Omega_c$ and therefore approach the ideal case of uniformly passing waves of frequencies $0 < \omega < \Omega_c$ without attenuation, and of uniformly rejecting all higher frequencies.

This might best be illustrated by the steady-state response of a low-pass filter with the voltage applied mid-series as shown in Fig. 3.3b. From (18), we take with $Z_s = 0$ the very simple general relations

$$V_q(p) = V_i(p)e^{-q\Gamma}, \quad I_q(p) = Y_c V_i(p)e^{-q\Gamma} \quad (70)$$

where Y_c is the reciprocal of the characteristic impedance and $V_i(p)$

the Laplace transform of the applied voltage. For a-c complex notation we have $V_i(p) = V/p - j\omega$ where $V = V_m e^{j\psi}$ is the complex amplitude (phasor); we find the steady-state response as the residue at $p = j\omega$, or explicitly

$$\bar{v}_q(t) = V(e^{-q\Gamma})_{p=j\omega} e^{j\omega t}; \quad \bar{i}_q(t) = V(Y_C e^{-q\Gamma})_{p=j\omega} e^{j\omega t} \quad (71)$$

The propagation function $\Gamma(\omega)$ follows from (43) and (44) with $p = j\omega$

$$\cosh \Gamma = 1 - \left(\frac{\omega}{\Omega_0}\right)^2, \quad \sinh \Gamma = j \frac{\omega}{\Omega_0} \sqrt{2 - \left(\frac{\omega}{\Omega_0}\right)^2} \quad (72)$$

If we define as in connection with (19)

$$\Gamma(\omega) = \alpha(\omega) + j\beta(\omega)$$

so that

$$\sinh \Gamma = \sinh \alpha \cos \beta + j \cosh \alpha \sin \beta$$

we must conclude from (72) that

$$\text{for } 0 < \omega < \left(\Omega_0 \sqrt{2} = \frac{2}{\sqrt{LC}}\right): \quad \alpha = 0, \quad \sin \beta = \frac{\omega}{\Omega_0} \sqrt{2 - \left(\frac{\omega}{\Omega_0}\right)^2} \quad (73)$$

$$\text{for } \omega > \Omega_0 \sqrt{2}: \quad \sinh \alpha = \frac{\omega}{\Omega_0} \sqrt{\left(\frac{\omega}{\Omega_0}\right)^2 - 2}, \quad \beta = \pi$$

or that all frequencies below $\Omega_c = \Omega_0 \sqrt{2}$, the *cutoff frequency*, are transmitted without any attenuation, and that all frequencies above Ω_c are attenuated with uniform phase shift $\beta = \pi$.

To obtain the actual voltage expression, we observe that

$$e^{-q\Gamma} = e^{-q\alpha(\omega)} e^{-jq\beta(\omega)}$$

Actually, if we define

$$\frac{\omega}{\Omega_c} = \sin \frac{\beta}{2} \quad \text{for } \omega \leq \Omega_c$$

$$\frac{\omega}{\Omega_c} = \cosh \frac{\alpha}{2} \quad \text{for } \omega \geq \Omega_c$$

we can rewrite (73) in the simpler forms

$$\text{for } 0 < \omega < \Omega_c: \quad \alpha = 0, \quad \beta = 2 \sin^{-1} \frac{\omega}{\Omega_c} \quad (73a)$$

$$\text{for } \omega > \Omega_c: \quad \alpha = 2 \cosh^{-1} \frac{\omega}{\Omega_c}, \quad \beta = \pi$$

and thus get

$$v_q(t) = \text{Im } \bar{v}_q(t) = \begin{cases} V_m \sin \left(\omega t + \psi - 2q \sin^{-1} \frac{\omega}{\Omega_c} \right) & \omega < \Omega_c \\ V_m e^{-2q \cosh^{-1} \frac{\omega}{\Omega_c}} \sin (\omega t + \psi - q\pi) & \omega > \Omega_c \end{cases} \quad (74)$$

Figure 3.6 gives a graphical representation of the relative voltage amplitude $\exp(-q\alpha)$ and phase angle $q\beta$ at the entrance of the q 'th section of the infinite filter. We see clearly, on the one hand, the sharp decline of the amplitude from its constant value beyond the

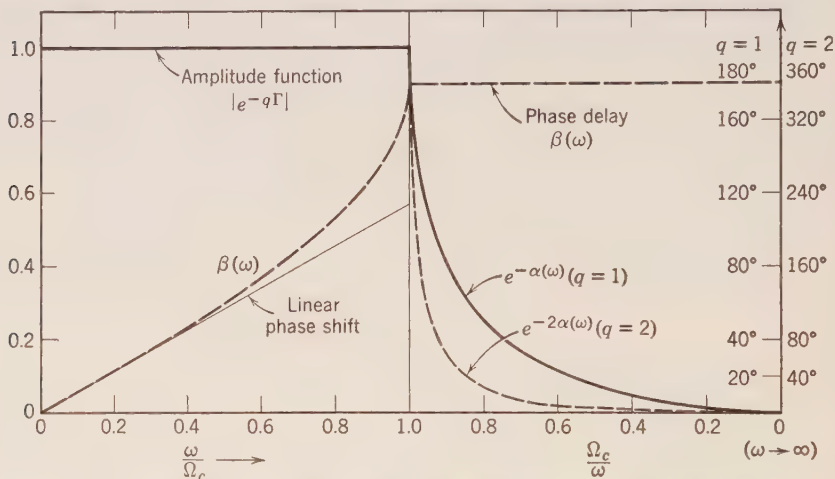


Fig. 3.6. Steady-state frequency characteristics of infinite low-pass filter giving amplitude $e^{-q\alpha(\omega)}$ and phase $q\beta(\omega)$ as functions of frequency.

cutoff frequency which is most indicative of the steady-state filter action. On the other hand, we need to note the strong variation of the phase angle for frequencies just below the cutoff frequency. Linearity of phase shift with frequency extends only to about $\omega \approx 0.4\Omega_c$.

For the current we also need the characteristic impedance, which follows from (45) as

$$Z_C = \frac{1}{j\omega C} (\sinh \alpha \cos \beta + j \cosh \alpha \sin \beta)$$

If we use the values of α and β from (73a) we can rationalize and obtain for the current

$$i_q(t) = \begin{cases} \frac{V_m}{\sqrt{L/C} \cos \beta/2} \sin (\omega t + \psi - q\beta) & \omega < \Omega_c \\ \frac{V_m}{\sqrt{L/C} \sinh \alpha/2} e^{-q\alpha} \sin (\omega t + \psi - q\pi) & \omega > \Omega_c \end{cases} \quad (75)$$

Because of the characteristic impedance variation, the current amplitude will vary with frequency monotonically, reaching infinite value (because of the lossless character of the filter) at the cutoff frequency.

To determine the transient characteristic of the infinite low-pass filter we first obtain from (43) and (44)

$$e^{+\Gamma} = \cosh \Gamma + \sinh \Gamma = 1 + \left(\frac{p}{\Omega_0}\right)^2 + \frac{p}{\Omega_0} \sqrt{2 + \left(\frac{p}{\Omega_0}\right)^2}$$

or if we introduce again the cutoff frequency $\Omega_c = \Omega_0 \sqrt{2} = 2/\sqrt{LC}$, we can combine this into the simpler form

$$e^{\Gamma} = (\sqrt{1+x^2} + x)^2, \quad x = \frac{p}{\Omega_c} \quad (76)$$

For the characteristic impedance we obtain from (45) with the same notation as above

$$Z_c = \frac{1}{pC} \sinh \Gamma = \sqrt{\frac{L}{C}} \sqrt{1+x^2} \quad x = \frac{p}{\Omega_c} \quad (77)$$

Let us find first the *indicial transmittance* of the infinite low-pass filter. We have to introduce into (70) $V_i(p) = V/p$ for the Laplace transform of the applied voltage and with (76) we obtain

$$\frac{V_a(p)}{V} = \frac{1}{\Omega_c} \frac{1}{x} (\sqrt{1+x^2} + x)^{-2a} \quad x = \frac{p}{\Omega_c} \quad (78)$$

The inverse Laplace transform can be adjusted for the scale factor Ω_c once we have the transform for the same function of p . Thus, if we have evaluated

$$\mathcal{L}f(t) = F(p) = \int_{0^+}^{\infty} f(t)e^{-pt} dt$$

and we want to find the corresponding transform involving $p/\Omega_c = x$, we note

$$F\left(\frac{p}{\Omega_c}\right) = \int_{0^+}^{\infty} f(t)e^{-\frac{p}{\Omega_c}t} dt = \Omega_c \int_{0^+}^{\infty} f(\Omega_c t)e^{-pt} dt$$

where the last form is simply the adjustment to the standard Laplace integral by taking $t = \Omega_c \tau$ and then letting $\tau \rightarrow t$. We therefore have the general relation

$$\mathcal{L}^{-1}F\left(\frac{p}{\Omega_c}\right) = \Omega_c f(\Omega_c t) \quad (79)$$

which has been entered in Table 1.4, line 6.

To find the inverse Laplace transform

$$\mathfrak{L}^{-1} \frac{1}{p} (\sqrt{1+p^2} + p)^{-2a} \quad (80)$$

we need to explore the Laplace integrals involving Bessel functions. An excellent approach is that given by Van der Pol-Bremmer,^{D15} chapter X.4, where it is demonstrated that

$$\mathfrak{L} J_n(t) = Q(p) = \frac{1}{\sqrt{p^2 + 1}} (\sqrt{p^2 + 1} + p)^{-n} \quad (81)$$

by transforming the differential equation of the Bessel function of the first kind $J_n(t)$ in the variable t

$$t^2 \frac{d^2 J_n}{dt^2} + t \frac{dJ_n}{dt} + (t^2 - n^2) J_n = 0 \quad n > 0 \quad (82)$$

into the Laplace transform domain. With the use of Table 1.4 for the transform of derivatives we find, because $J_n(0) = 0$ for $n > 0$

$$\mathfrak{L} J_n(t) = Q(p), \quad \mathfrak{L} \frac{d}{dt} J_n(t) = pQ(p)$$

$$\mathfrak{L} \frac{d^2}{dt^2} J_n(t) = p^2 Q(p)$$

Since $J_n(t)$ is a continuous function in t , we can differentiate its Laplace integral with respect to p , namely

$$\frac{d}{dp} \int_{0^+}^{\infty} J_n(t) e^{-pt} dt = - \int_{0^+}^{\infty} t J_n(t) e^{-pt} dt = -\mathfrak{L} t J_n(t) \quad (83)$$

and do the same for its derivatives. We thus have obtained the necessary transforms for each term in (82) and can write the transform of the entire equation

$$\frac{d^2}{dp^2} (p^2 Q) - \frac{d}{dp} (pQ) + \left(\frac{d^2}{dp^2} - n^2 \right) Q = 0$$

Upon differentiating and collecting terms we arrive at the transform differential equation

$$(p^2 + 1) \frac{d^2 Q}{dp^2} + 3p \frac{dQ}{dp} + (1 - n^2) Q = 0 \quad (84)$$

which has the algebraic solution $Q(p)$ given in (81) as can be readily verified by substitution. A different demonstration involving the

TABLE 3.1
LAPLACE TRANSFORMS OF BESSEL FUNCTIONS AND RELATED FUNCTIONS

No.	Time Function	Laplace Transform	Remarks
1	$\omega J_n(\omega t)$	$\frac{1}{\sqrt{x^2 + 1}} (\sqrt{x^2 + 1} + x)^{-n}$	$x = \frac{p}{\omega}$
2	$\omega J_0(\omega t)$	$\frac{1}{\sqrt{x^2 + 1}}$	$x = \frac{p}{\omega}$
3	$\omega J_1(\omega t)$	$\frac{1}{\sqrt{x^2 + 1}} (\sqrt{x^2 + 1} + x)^{-1}$	$x = \frac{p}{\omega}$
4	$\omega \frac{J_n(\omega t)}{\omega t}$	$\frac{1}{n} (\sqrt{x^2 + 1} + x)^{-n}$	$x = \frac{p}{\omega}$
11	$\omega I_n(\omega t)$	$\frac{1}{\sqrt{x^2 - 1}} (\sqrt{x^2 - 1} + x)^{-n}$	$x = \frac{p}{\omega}$
12	$\omega I_0(\omega t)$	$\frac{1}{\sqrt{x^2 - 1}}$	$x = \frac{p}{\omega}$
13	$\omega I_1(\omega t)$	$\frac{1}{\sqrt{x^2 - 1}} (\sqrt{x^2 - 1} + x)^{-1}$	$x = \frac{p}{\omega}$
14	$\omega I_n \left(\omega \frac{t}{2} \right)$	$\frac{2^{n+1}}{\sqrt{2x - 1} \sqrt{2x + 1}} (\sqrt{2x - 1} + \sqrt{2x + 1})^{-2n}$	Scale factor applied to 11
15	$\omega I_n \left(\omega \frac{t}{2} \right) e^{-\frac{\omega t}{2}}$	$\frac{1}{\sqrt{x} \sqrt{x + 1}} (\sqrt{x} + \sqrt{x + 1})^{-2n}$	Shifting theorem applied to 14
16	$n \frac{I_n \left(\omega \frac{t}{2} \right)}{\omega t} e^{-\frac{\omega t}{2}}$	$(\sqrt{x} + \sqrt{x + 1})^{-2n}$	Integration 26 applied to 15
General Formal Relations:			
21	$f(t)$	$F(p)$	Basic relation
22	$tf(t)$	$-\frac{d}{dp} F(p)$	(3.83) deduction
23	$\omega f(\omega t)$	$F(x)$	$x = \frac{p}{\omega}$
24	$\omega \omega t f(\omega t)$	$-\frac{d}{dx} F(x)$	$x = \frac{p}{\omega}$
25	$\frac{1}{t} f(t)$	$\int_p^\infty F(p) dp$	(3.114) deduction
26	$\frac{1}{\omega} \frac{1}{\omega t} f(\omega t)$	$\int_x^\infty F(x) dx$	$x = \frac{p}{\omega}$

series expansion of the basic form $(p^2 + 1)^{-1/2}$ is given in Gardner-Barnes,^{D3} pp. 317-319; in McLachlan,^{D13} pp. 137-142; and in Doetsch,^{D5} p. 48.* The Laplace transform (81) is listed in Table 3.1 as basic for Bessel functions. Though the preceding demonstration does not hold directly for $n = 0$, one can deduce from the relation for $J_1(t)$ that $n = 0$ in (81) gives actually the transform of $J_0(t)$, which is also listed in Table 3.1 for later use.

* See also G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, England, 1922; N. W. McLachlan, *Bessel Functions for Engineers*, Oxford University Press, England, 1934.

Returning now to (80), we can surely interpret the factor $1/p$ by Table 1.4 as integration. If we designate the form.

$$(\sqrt{1 + p^2} + p)^{-n} = \mathcal{L}g(t)$$

and differentiate with respect to p , obtaining on the right-hand side by (83) the Laplace transform of $[-tg(t)]$, we find

$$\frac{d}{dp} \mathcal{L}g(t) = -\frac{n}{\sqrt{p^2 + 1}} (\sqrt{p^2 + 1} + p)^{-n} = -\mathcal{L}tg(t)$$

But the center term is identical with $Q(p)$ in (81), so that we deduce

$$g(t) = \mathcal{L}^{-1}(\sqrt{1 + p^2} + p)^{-n} = n \frac{J_n(t)}{t} \quad (85)$$

which we also list in Table 3.1. Now we can write for (80)

$$\mathcal{L}^{-1} \frac{1}{p} (\sqrt{1 + p^2} + p)^{-n} = n \int_{0^+}^t \frac{J_n(t)}{t} dt$$

Since (78) has the scale factor Ω_c incorporated, we must make the adjustment defined in (79) which leads to the final solution

$$\frac{v_q(t)}{V} = 2q \int_{y=0}^{y=\Omega_c t} J_{2q}(y) \frac{dy}{y} \quad (86)$$

which can be evaluated only by numerical integration, but nevertheless represents an extremely simple formal solution.

For the *indicial admittance* of this infinite low-pass filter we take from (70) with the explicit forms (76) and (77)

$$\frac{I_q(p)}{V} = \frac{1}{p} \frac{1}{\sqrt{L/C}} \frac{(\sqrt{1 + x^2} + x)^{-n}}{\sqrt{1 + x^2}} \quad (87)$$

Here we can read the inverse Laplace transform of the last factor directly from (81) or from Table 3.1. The complete solution is then

$$\frac{i_q(t)}{V} = \frac{1}{\sqrt{L/C}} \int_{y=0}^{y=\Omega_c t} J_{2q}(y) dy \quad (88)$$

where we used $y = \Omega_c t$ to simplify the writing. Figure 3.7 shows the values of the integral for $q = 0, 3$, and 5 , as a function of the upper limit $\Omega_c t$. Note that the current remains very small until $\Omega_c t \approx 2q$, then increases rapidly and reaches its final steady-state value in an oscillatory manner, the frequency of this oscillation slowly increasing and approaching Ω_c as can be seen from the asymptotic expression of

the Bessel function

$$J_{2q}(y) \approx \sqrt{\frac{2}{\pi y}} \cos\left(y - \frac{4q+1}{4}\pi\right) \quad y \gg (4q)^2 \quad (89)$$

It is important to recognize the delay in build-up as an *apparent* but not actual propagation effect inasmuch as the solution (88) has finite

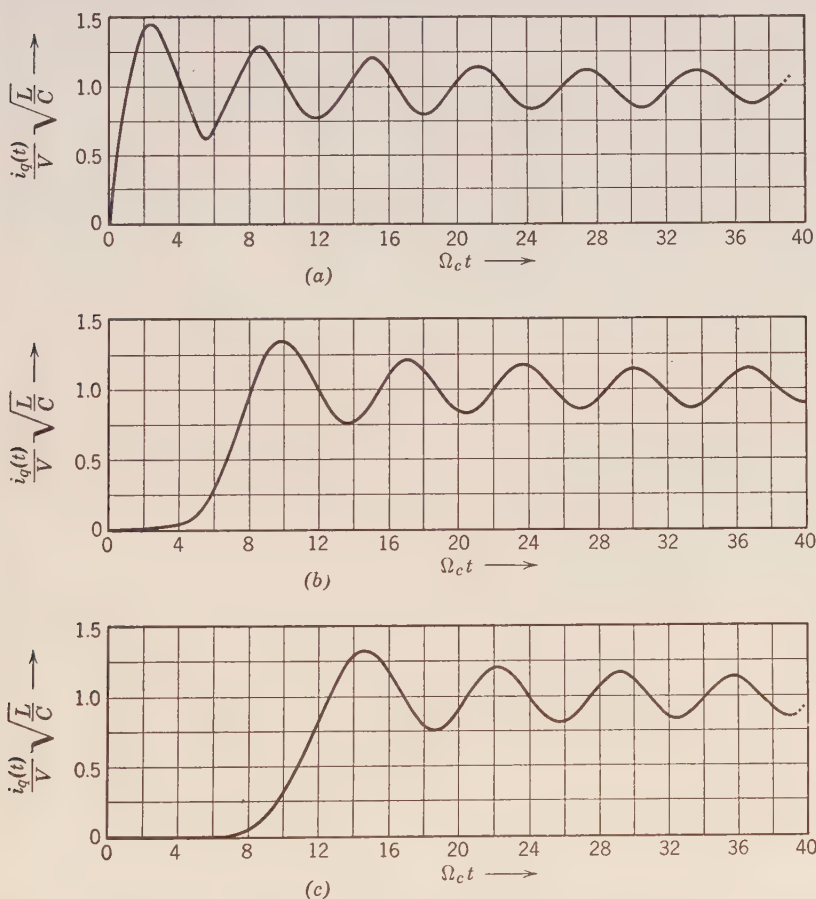


Fig. 3.7. Response of infinite low-pass filter to a step voltage; indicial admittance (a) in first section, (b) in third section, (c) in fifth section. Reproduced from *Bell System Tech. J.*, July 1923, with permission of the publisher.

values for any time $t > 0$ in any one of the sections q however large q might be. This is, of course, because we have disregarded any spatial extension of the fourpole line and thus have assumed instantaneous action of the applied source function all along the network.

A very detailed discussion of this solution, as well as the solutions for high-pass and band-pass filters with many graphs, can be found in the original and classical treatise by Carson and Zobel* and is partly reproduced in Carson.†

It is interesting to attempt the solution for the building-up of sinusoidal currents. If we apply a sinusoidal voltage, using complex notation with the Laplace transform $V_m/p - j\omega$, the transform solution for the current in (87) will change to

$$\bar{I}_q(p) = \frac{V_m}{p - j\omega} \frac{1}{\sqrt{L/C}} \frac{(\sqrt{1+x^2}+x)^{-n}}{\sqrt{1+x^2}} \quad (90)$$

Here we can apply the special form of the Borel theorem from Table 1.5, line 12, since we know the inverse transform of the last factor from (81) and we obtain in complex notation

$$\bar{i}_q(t) = \frac{V_m}{\sqrt{L/C}} e^{j\omega t} \int_{\eta=0}^{\eta=\Omega_c t} e^{-j\lambda\eta} J_{2q}(\eta) d\eta$$

where we substituted $\eta = \Omega_c t$ and $\lambda = \omega/\Omega_c$. If we take the imaginary part only of this form, we need to combine the outside exponential with that inside the integral. To avoid its involvement in the integration process, we shall designate it as $e^{j\lambda y}$ where again $y = \Omega_c t$ but is considered a fixed parameter for the integration. Thus

$$i_q(t) = \frac{V_m}{\sqrt{L/C}} \int_{\eta=0}^{\eta=y} J_{2q}(\eta) \sin \lambda (y - \eta) d\eta = \frac{V_m}{\sqrt{L/C}} S_{2q}(y, \lambda) \quad (91)$$

The integral $S_n(y, \lambda)$ is of a standard type and is discussed thoroughly in the treatise by Carson and Zobel, *loc. cit.* The following approximations bring out the important features of the solution:

(a) For $\lambda < 1$, or $\omega < \Omega_c$, i.e., for sine waves of frequencies *within the pass band*, the total solution approaches for $y = \Omega_c t > 2q/\sqrt{1-\lambda^2}$ the form

$$S_{2q}(y, \lambda) = \frac{\lambda}{\lambda^2 - \sqrt{1 - (2q/y)^2}} J_{2q}(y) + \frac{1}{\sqrt{1 - \lambda^2} \sin(\omega t - 2q \sin^{-1} \lambda)} \quad (92)$$

* J. R. Carson and O. J. Zobel, "Transient Oscillations in Electric Wave Filters," *Bell System Tech. J.*, 1-52 (July 1923); see also J. R. Carson, "Theory of the Transient Oscillations of Electric Network and Transmission Systems," *Trans. AIEE*, **38**, 345-427, particularly 375, and Figs. 1-4 (1919).

† J. R. Carson, *Electric Circuit Theory and Operational Calculus*, McGraw-Hill, New York, 1926.

It represents the superposition of the steady-state response to the applied frequency ω and the decaying transient oscillation of character similar to that in the indicial admittance (88) and with the same approximation (89). Figure 3.8a shows the results for $q = 5$ and

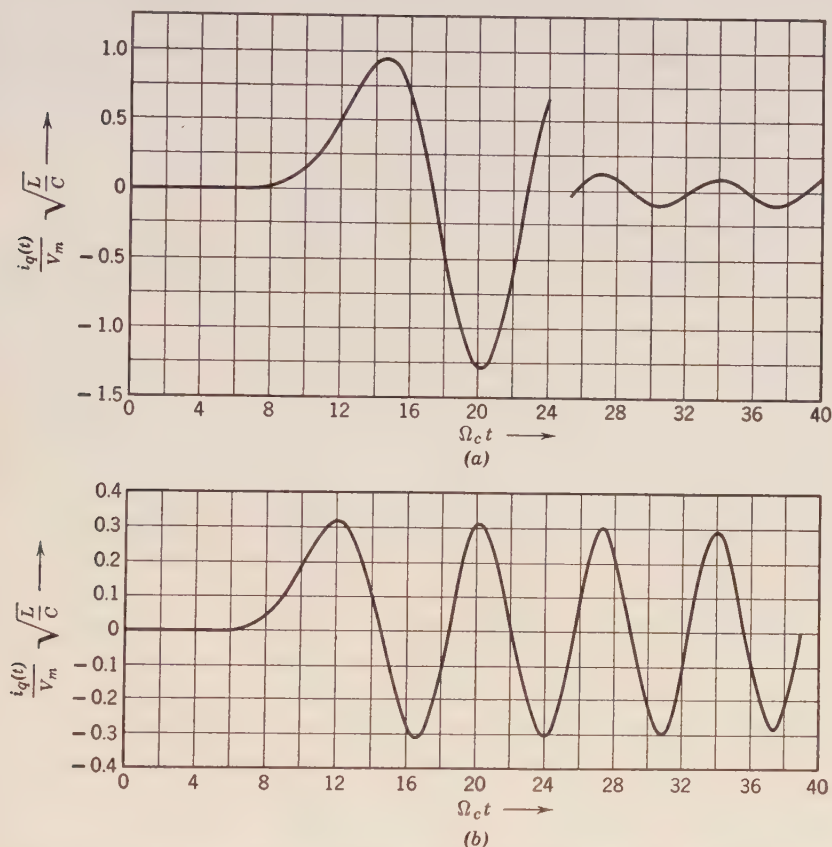


Fig. 3.8. Response of infinite low-pass filter to sinusoidal voltage; current in fifth section (a) for applied frequency $\omega = \frac{1}{2} \Omega_c$, (b) for applied frequency $\omega = 1.25 \Omega_c$. Reproduced from *Bell System Tech. J.*, July 1923, with permission of the publisher.

$\lambda = \frac{1}{2}$, giving for $\Omega_c t > 25$ only the difference between the total response and the final steady-state values.

(b) For $\lambda > 1$, or $\omega > \Omega_c$, i.e., for sine waves of frequencies outside the pass band, the final solution approaches for $y = \Omega_c t > 2q$ the form

$$S_{2q}(y, \lambda) = \frac{\lambda}{\lambda^2 - \sqrt{1 - (2q/y)^2}} J_{2q}(y) \quad (93)$$

Even though the filter will respond with only an extremely small steady-state oscillation of the impressed frequency ω , its transient response will be very significant and depends primarily upon the ratio $\lambda = \omega/\Omega_c$. Figure 3.8*b* shows the response for $q = 5$ and $\lambda = 1.25$. Obviously, the selectivity of the filter does not extend to the transient response! This is a very important fact for the practical design of selective systems.

(c) The build-up time of the sine waves of frequency ω can be estimated as

$$\frac{2q}{\Omega_c} \frac{1}{\sqrt{1 - (\omega/\Omega_c)^2}}$$

where $2q/\Omega_c$ represents the delay caused by the apparent propagation effect as in the indicial admittance, and where the second factor gives the effect of the ratio of applied to cutoff frequencies.

Similar results hold for the other types of conventional filters, the only ones for which computations have been made. Extensive discussions of high-pass and band-pass filter performance under transient conditions are given in Carson-Zobel, *loc. cit.* A few simple examples are also given in McLachlan,^{D13} pp. 224-237. Van der Pol and Bremmer,^{D15} p. 194, give the response of a non-homogeneous low-pass filter.

3.8 Analogues

Many mechanical, acoustic, or thermal systems exhibit characteristics very similar to those of electric wave filters and can be treated by the same mathematical methods. In fact, as stressed in Vol. I, chapter 3, these systems can be studied by the corresponding electrical analogue which in many instances leads to a simple analogue computer. For the sake of convenience, Table 3.2 reproduces the analogous quantities in systems with one degree of freedom which can readily be extended to more complicated systems.

Torsional vibrations. As a simple example we might examine the *torsional vibrations of a crankshaft*, carrying a load on one end, a flywheel on the opposite end, and subjected to N equally spaced driving cranks which might be represented as discs* as indicated in Fig. 3.9. Disregarding the weight of the crankshaft itself and considering it only to provide ideal coupling between the rotating discs, the flywheel, and the load, we might take its angular displacement at disc q as θ_q . The resulting elastic torque of the portion of the crankshaft

* J. P. Den Hartog, *Mechanical Vibrations*, chapter V, p. 234, McGraw-Hill, New York, 1947.

TABLE 3.2
ANALOGOUS SYSTEMS WITH ONE DEGREE OF FREEDOM

Quantity	Mechanical				Thermal System
	Electric Circuit	Translational System	Rotational System	Acoustic System	
Generalized coordinate	Charge q	Displacement x	Angular displacement θ	Displacement x	Quantity of heat Q
Generalized velocity	Current $i = \dot{q}$	Velocity $u = \dot{x}$	Angular velocity $\Omega = \dot{\theta}$	Velocity $u = \dot{x}$	Heat power $\dot{q} = \frac{dQ}{dt}$
Generalized force	Voltage $v(t)$	Force $F(t)$	Torque $T(t)$	Pressure (total) $P(t)$	Temperature T
Kinetic parameter	Inductance L	Mass m	Moment of inertia J	Fluid mass M	
Potential parameter	Elastance $S = 1/C$	Stiffness k	Rotational stiffness K	Fluid stiffness S_f	
Inverse potential parameter	Capacitance C	Compliance $1/k = C_m$	Rotational compliance $1/K = C_r$	Fluid compliance C_f	Thermal capacitance C_{th}
Dissipation parameter	Resistance R	Friction coefficient f	Friction coefficient B	Acoustic resistance R_f	Thermal resistance R_{th}
		also R_M	also R_r		
Potential energy U	Conductance $G = 1/R$	Motional resistance R_M	Rotational resistance R_r	Acoustic mobility $G_f = 1/R_f$	Thermal conductance $G_{th} = 1/R_{th}$
Kinetic energy W	$\frac{1}{2} \frac{1}{C} q^2$	Mobility $G_M = 1/R_M$	Rotational mobility $G_r = 1/B$	$\frac{1}{2} S_f x^2$	
Dissipative power $2D$	$\frac{1}{2} Li^2$	$\frac{1}{2} kx^2$	$\frac{1}{2} K\theta^2$	$\frac{1}{2} Mu^2$	
Complex notation for sinusoidal oscillations	Impedance $Z(j\omega) = V/I$	Motional impedance $Z_M(j\omega) = F/u$	Rotational impedance $Z_r(j\omega) = I/\Omega$	Acoustic (specific) impedance $Z_a(j\omega) = p/u$	
	Admittance $Y(j\omega) = I/V$	Complex mobility $Y_M(j\omega) = u/F$	Complex rotational mobility $Y_r(j\omega) = \Omega/I$		

between discs q and $q + 1$ acting to the right on q will have the magnitude

$$K(\theta_q - \theta_{q+1})$$

and similarly $K(\theta_{q-1} - \theta_q)$ will act on the left of disc q ; K is the rotational (torsional) stiffness of the shaft. The difference of these two

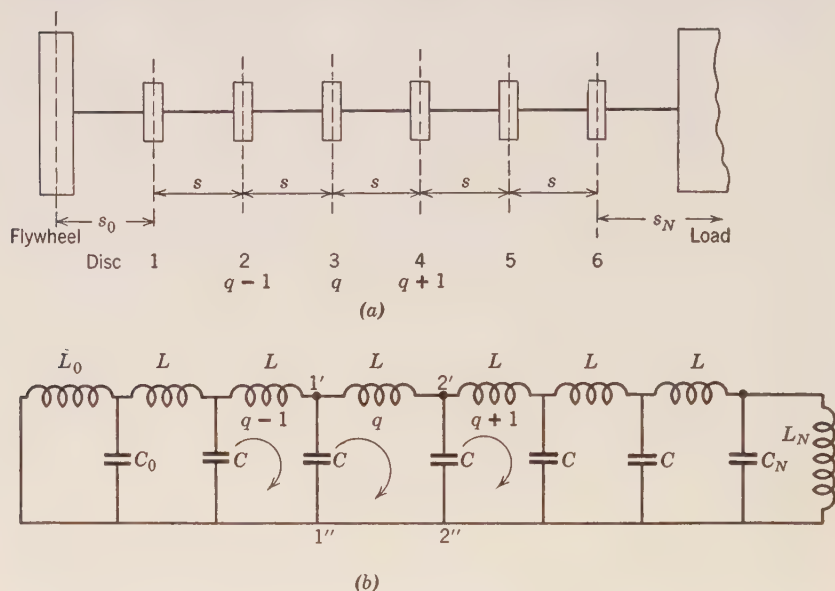


Fig. 3.9. Schematic representation of a multicylinder engine: (a) mechanical schematic, (b) electrical analogue.

elastic torques acts accelerating upon the disc q , and if we assume its moment of inertia to be J , we have

$$J \frac{d^2\theta_q}{dt^2} = K(\theta_{q-1} - \theta_q) - K(\theta_q - \theta_{q+1}) \quad (94)$$

This corresponds to the relationship between the node-pair voltages $1'1''$ and $2'2''$ if we use the mesh currents i_q , etc., in the respective meshes. We would have in accordance with the analogue in Fig. 3.9b

$$L \frac{di_q}{dt} = \frac{1}{C} \int_0^t (i_{q-1} - i_q) dt - \frac{1}{C} \int_0^t (i_q - i_{q+1}) dt$$

or upon differentiating

$$L \frac{d^2i_q}{dt^2} = \frac{1}{C} (i_{q-1} - 2i_q + i_{q+1}) \quad (95)$$

Introducing Laplace transforms

$$\mathcal{L}i_q(t) = I_q(p)$$

we have for (95)

$$I_{q-1}(p) - (2 + p^2 LC)I_q(p) + I_{q+1}(p) = 0 \quad (96)$$

which is the basic four pole line equation (5) with

$$\mathfrak{A} = \mathfrak{D} = 1 + \frac{1}{2}p^2 LC$$

This is to be expected for the low-pass filter represented by Fig. 3.9b and all the results of section 3.4 apply. In particular, if we choose short-circuit terminations at both ends of the six uniform sections, which are equivalent to mechanically "free" ends of the crankshaft, the critical frequencies* are given directly by (48) with the cutoff frequency $\Omega_c = 2/\sqrt{LC}$. If the actual terminations are maintained, the complete solution must be invoked, which is rather involved. In this case the critical frequencies can be obtained only by a graphical plot; see Thomson,^{D18} pp. 172-174. Conversely, mechanical filters can be designed with specified frequency characteristics† in completely analogous manner to the electric wave filters.

Coupled moving masses. A train made up of N like cars each of mass m and with elastic coupling coefficients $k_{\alpha\beta}$ and damping dashpots $f_{\alpha\beta}$ between any two successive cars represents a delicate mechanical structure that can easily exhibit longitudinal oscillations of rather unpleasant magnitudes.‡ If we are particularly interested in the requirements we must satisfy to avoid these oscillations, we can evaluate the natural modes of response without carrying through a complete solution. The electrical analogue can readily be constructed by appropriately extending the system for two coupled masses shown in Vol. I, Fig. 3.7. Thus, Fig. 3.10a shows the schematic mechanical arrangement, where the couplings are assumed all equal. The rolling friction on the track is indicated by the coefficient f . The electrical analogue is given in Fig. 3.10b, where the coupling capacitances and conductances are exactly $(N - 1)$ in number, and the inductances L representing the cars, with R their rolling friction, are N in number. This has again the appearance of a low-pass filter but with consider-

* Karman and Biot, *op. cit.*, p. 459. Also Thomson,^{D18} pp. 170-172.

† W. P. Mason, *Electrodynamical Transducers and Wave Filters*, Van Nostrand, New York, 1942. Also Karman-Biot, *op. cit.*, pp. 461-465. Also R. Adler, "Compact Electromechanical Filter," *Electronics*, **20**, 100 (1947).

‡ L. A. Pipes, "Analysis of Longitudinal Motion of Trains by the Electrical Analogue," *J. Appl. Phys.*, **13**, 780-786 (1942).

able dissipation. Choosing as the individual fourpole the T-section in Fig. 3.10c, we note that the fourpole line has $(N - 1)$ fourpoles and is terminated at either end into $Z/2$ if $Z = R + pL$ is the series impedance element. We disregard all external sources.

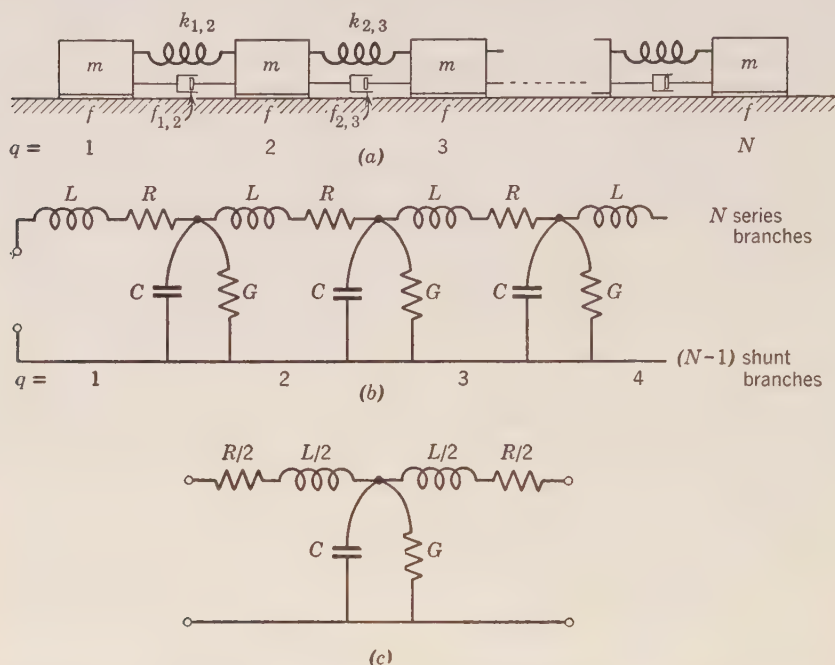


Fig. 3.10. Schematic representation of a train of N like cars: (a) mechanical schematic, (b) electrical analogue, (c) the individual fourpole.

The characteristic equation for the natural modes of response has to be taken from the general solutions (16) or (17) and is defined by the vanishing of the denominator, or

$$e^{2(N-1)\Gamma} = \rho_S \rho_T \quad (97)$$

We need to replace N in (16) by $(N - 1)$. The characteristic impedance of the fourpole line is from (42)

$$Z_C = \left(\frac{Z}{Y}\right)^{1/2} \left(1 + \sinh^2 \frac{\Gamma}{2}\right)^{1/2} = \left(\frac{Z}{Y}\right)^{1/2} \cosh \frac{\Gamma}{2}$$

The reflection coefficients at both terminations are the same, namely

$$\rho_S = \rho_T = \frac{\frac{1}{2}Z - Z_C}{\frac{1}{2}Z + Z_C} = \frac{\frac{1}{2}\sqrt{YZ} - \cosh(\Gamma/2)}{\frac{1}{2}\sqrt{YZ} + \cosh(\Gamma/2)}$$

However, again from (42) we take $\frac{1}{2} \sqrt{YZ} = \sinh (\Gamma/2)$, so that

$$\rho_S = \rho_T = -e^{-\Gamma}$$

and therefore (97) becomes

$$e^{2(N-1)\Gamma} = e^{-2\Gamma}, \quad e^{2N\Gamma} = 1 \quad (98)$$

This result is *completely general*, and applies for any fourpole line composed of N complete series elements and $(N - 1)$ complete shunt elements, short-circuited at both ends. The values of Γ satisfying (98) are

$$2N\Gamma = j\alpha 2\pi$$

$$\Gamma_\alpha = j \frac{\alpha\pi}{N}, \quad \alpha = 1, 2, \dots, N \quad (99)$$

If we first disregard the rolling friction, $R = 0$, we have

$$Z = pL, \quad Y = G + pC$$

and therefore, again from (42)

$$\cosh \Gamma = 1 + \frac{1}{2}YZ = 1 + \frac{1}{2}pL(G + pC)$$

from which to determine the root values p_α which are the actual natural modes or "free" modes of response. With Γ_α from (99), we find now

$$\frac{1}{2}p^2LC + \frac{1}{2}pLG + 1 - \cos \frac{\alpha\pi}{N} = 0$$

a quadratic relation for each value α , so that we obtain

$$\left. \begin{matrix} p_{\alpha'} \\ p_{\alpha''} \end{matrix} \right\} = -\frac{G}{2C} \pm \left[\left(\frac{G}{2C} \right)^2 - \frac{2}{LC} (1 - \cos \theta_\alpha) \right]^{1/2}$$

where $\theta_\alpha = \alpha\pi/N$. To avoid oscillatory response to any disturbance of the mechanical system, from whatever cause, we must require that all $p_{\alpha'}$, $p_{\alpha''}$ be real and negative. This in turn requires that the radicand remain positive for all values of θ . The largest values that the second term can assume will be $4/LC$ if $\theta_\alpha = \pi$, so that the condition of no oscillatory mode of response becomes simply

$$\left(\frac{G}{2C} \right)^2 > \frac{4}{LC} \quad G > 4 \sqrt{C/L} \quad (100)$$

If we now reinstate the rolling friction, we have

$$Z = R + pL, \quad Y = G + pC$$

and the solution for the natural modes becomes correspondingly,

$$\left. \begin{matrix} p_{\alpha'} \\ p_{\alpha''} \end{matrix} \right\} = -\delta \pm \left(\delta^2 - \frac{RG}{LC} - \frac{2}{LC} (1 - \cos \theta_a) \right)^{1/2} \quad (101)$$

where we used

$$\delta = \frac{1}{2} \left(\frac{G}{C} + \frac{R}{L} \right)$$

as the total damping coefficient of the system. We find

$$\delta^2 - \frac{RG}{LC} = \frac{1}{4} \left(\frac{G}{C} - \frac{R}{L} \right)^2$$

so that the condition of nonoscillatory response becomes

$$G > 4 \sqrt{C/L} + 2 \frac{RC}{L} \quad (102)$$

The rolling friction tends to promote the longitudinal oscillation of the system! The existence of rolling friction requires stronger damping in the couplings in order to prevent oscillatory motion, a very important result obtained from the simple analogy.

Quasi-distortionless fourpole line. The general fourpole line of Fig. 3.10*b* has the modes of response given in (101). If we define a quantity

$$\sigma = \frac{1}{2} \left(\frac{G}{C} - \frac{R}{L} \right) \quad (103)$$

we observe that for $\sigma = 0$, or $G/C = R/L$, these modes reduce to

$$\left. \begin{matrix} p_{\alpha'} \\ p_{\alpha''} \end{matrix} \right\} = -\delta \pm j\Omega_0 \left(1 - \cos \frac{\alpha\pi}{N} \right)^{1/2} \quad (104)$$

where $\Omega_0^2 = 2/LC$. Comparing these expressions with (47a) we see that the natural frequencies of (104) are identical with those of the lossless low-pass filter, i.e., that the fourpole line with $\sigma = 0$ possesses the same spectral behavior as the lossless line, i.e., has *distortionless* character. We call σ the distortion constant, and a line for which $\sigma = 0$, *quasi-distortionless*. Such a line possesses considerable attenuation

$$\sigma = 0, \quad \delta = \frac{G}{C} = \frac{R}{L}$$

which, however, is independent of frequency, so that simple adjustments can be made to take damping into consideration. For exam-

ple, the indicial admittance of the lossless low-pass infinite wave filter was given in (88). We have now, respectively, for the

lossless wave filter

$$Y = pC$$

$$Z = pL$$

distortionless wave filter

$$Y' = pC + G = C(p + \delta)$$

$$Z' = pL + R = L(p + \delta)$$

so that replacement of p by $(p + \delta)$ will lead to the complete solution for the quasi-distortionless line. This adjustment is readily made by the shifting theorem of Table 1.4, line 5. We thus can write

$$\frac{i_q(t)}{V} = \frac{1}{\sqrt{L/C}} \int_{t=0}^t e^{-\delta t} J_{2q}(\Omega_c t) d(\Omega_c t) \quad (105)$$

for the indicial admittance of the quasi-distortionless line. Lines of this type have particular importance for measurement purposes, e.g., for high-voltage surges.*

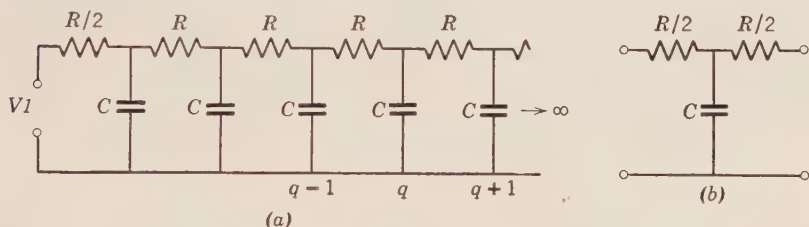


Fig. 3.11. Infinite R - C fourpole line: (a) step voltage applied at mid-series point, (b) individual fourpole.

Thermal systems. Because thermal phenomena involve only one kind of energy, no oscillatory responses can occur and the electrical analogue will involve only resistance and capacitance branches;† see also Vol. I, section 3.6. Suppose we examine heat diffusion in a very extended thermal system which we might represent by the infinitely long R - C fourpole line shown in Fig. 3.11a with the individual fourpole in b. We have here for series and shunt element, respectively,

$$Z = R, \quad Y = pC$$

so that from (42), or even better, (26) we get

$$\cosh \Gamma = 1 + 2x, \quad e^\Gamma = 1 + 2x + 2\sqrt{x}\sqrt{1+x}$$

* C. L. Dawes, C. M. Thomas, and A. B. Drought, "Impulse Measurements by Repeated-Structure Networks," *Trans. AIEE*, **69**, Part I, 571-583 (1950).

† J. H. Neher, "The Determination of Temperature Transients in Cable Systems by Means of an Analog Computer," *Trans. AIEE*, **70**, Part II, 1361-1370 (1951).

if we abbreviate

$$x = \frac{1}{4}RCp = pT$$

where T might be taken as a basic time constant. Actually, we can identify

$$e^x = (\sqrt{x} + \sqrt{1+x})^2 \quad (106)$$

as squaring readily indicates. Similarly, we can express the characteristic impedance from (42) or (27) as

$$Z_c = Z \frac{1}{\sqrt{YZ}} \left(1 + \frac{1}{4}YZ\right)^{-\frac{1}{2}} = \frac{1}{2}R \frac{\sqrt{1+x}}{\sqrt{x}} \quad (107)$$

Since voltage is the analogue to temperature and current the analogue to heat power, we might pose the problem: applying a constant temperature at the input end of the uniform thermal system, what is the distribution of heat power throughout the system as a function of time. In the analogous problem it requires the solution of the current distribution if at time $t = 0$ a step voltage VI is applied at the mid-series point as indicated in Fig. 3.11a. The general solution is given by (70), or if we use (106) and (107)

$$V_q(p) = \frac{V}{p} (\sqrt{x} + \sqrt{1+x})^{-2q} \quad (108)$$

$$I_q(p) = \frac{2VT}{R} \frac{1}{\sqrt{x} \sqrt{1+x}} (\sqrt{x} + \sqrt{1+x})^{-2q} \quad (109)$$

The inverse Laplace transform of (109) can be deduced from a previously established form, namely line 11 of Table 3.1

$$\mathcal{L}\omega I_n(\omega t) = \frac{1}{\sqrt{x^2 - 1}} (\sqrt{x^2 - 1} + x)^{-n}$$

where $I_n(\omega t)$ is the modified Bessel function of the first kind.* If we replace t by $t/2$ we can use the general relation for a scale factor, line 6 in Table 1.4, namely replacing on the right-hand side x by $2x$ and attaching a factor 2 to the whole form. We can further identify

$$(\sqrt{4x^2 - 1} + 2x)^{-n} = [\frac{1}{2}(\sqrt{2x - 1} + \sqrt{2x + 1})^2]^{-n}$$

so that we establish

$$\mathcal{L}\omega I_n\left(\omega \frac{t}{2}\right) = 2^{n+1} \frac{(\sqrt{2x - 1} + \sqrt{2x + 1})^{-2n}}{\sqrt{2x - 1} \sqrt{2x + 1}} \quad (110)$$

* Karman-Biot, *loc. cit.*; also McLachlan,^{D13} and N. W. McLachlan, *loc. cit.*

which is entered in line 14 of Table 3.1 because of its more general significance.

If we now also apply to the time function the exponential factor $\exp (-\omega t/2)$, we need to replace p by $[p+(\omega/2)]$ or $2x$ by $(2x+1)$. This leads at once to the relation

$$\mathcal{L}\omega I_n\left(\omega\frac{t}{2}\right)e^{-\omega\frac{t}{2}}=\frac{(\sqrt{x}+\sqrt{1+x})^{-2n}}{\sqrt{x}\sqrt{1+x}}\tag{111}$$

which is exactly what we need for (109) except that in Table 3.1 we

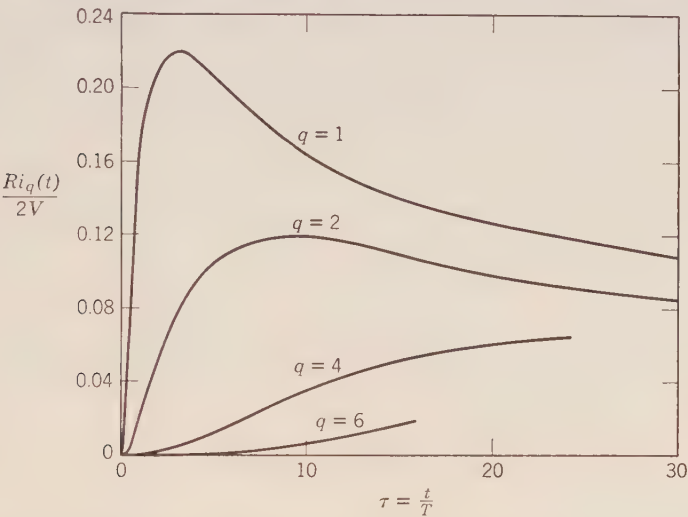


Fig. 3.12. Heat diffusion along infinite cable represented by R - C fourpole line; indicial admittance at entrance of first section $q=1$, second section $q=2$, fourth section $q=4$, and sixth section $q=6$.

defined $x=p/\omega$ whereas in (109) we used $x=pT$. With this modification we thus find

$$i_q(t)=\frac{2V}{R}I_q\left(\frac{t}{2T}\right)e^{-\frac{t}{2T}}\tag{112}$$

Figure 3.12 shows this diffusion process at the input terminals of several T -sections, namely $q=1, 2, 4, 6$. For values of the argument large compared with the order q of the modified Bessel function we can use the asymptotic expression

$$\lim_{y\gg q}I_q(y)\rightarrow\frac{e^y}{\sqrt{2\pi y}}$$

so that

$$i_q(t) \rightarrow \frac{2V}{R} \sqrt{\frac{T}{2t}} = V \sqrt{\frac{C}{\pi R t}} \quad (113)$$

To obtain the inverse Laplace transform of (108) we might observe that

$$\frac{d}{dx} (\sqrt{x} + \sqrt{1+x})^{-2q} = \frac{q}{\sqrt{x} \sqrt{1+x}} (\sqrt{x} + \sqrt{1+x})^{-2q}$$

so that we might apply directly line 24 from Table 3.1.

Conversely, we could apply integration of the Laplace transform $F(p)$ with respect to p

$$\int_p^\infty F(p) dp = \int_p^\infty \left[\int_0^\infty f(t) e^{-pt} dt \right] dp$$

where $f(t)$ is the inverse Laplace transform to $F(p)$ by assumption. We can therefore interchange the integrations so that we obtain

$$\int_0^\infty \left(\int_p^\infty e^{-pt} dp \right) f(t) dt = \int_0^\infty \left(\frac{1}{t} e^{-pt} \right) f(t) dt$$

This, however, is the direct Laplace transform of $f(t)$ divided by t , so that we have the general relation

$$\mathfrak{L} \frac{1}{t} f(t) = \int_p^\infty F(p) dp \quad (114)$$

which is also entered in Table 3.1 as line 25. Applied to (111) we find the relation given in line 16 of Table 3.1, and thus have for the voltage as the inverse Laplace transform of (108)

$$v_q(t) = qV \int_{\theta=0}^{\theta} \frac{1}{\theta} e^{-\theta} I_q(\theta) d\theta \quad (115)$$

where $\theta = t/2T$. This integral requires numerical or graphical integration.

PROBLEMS

3.1 Take the cascade of four R - C sections as in Fig. 3.2. (a) Apply the mesh equations of Kirchhoff; (b) apply the method of "fourpole lines" and demonstrate that the Laplace transform solutions for the output current are identical. Assume a step voltage $V_m 1$ applied directly at the sending end and short-circuit the far end.

3.2 Find the output voltage of the finite R - C line if the far end is open-circuited and if we apply the step voltage $V_m 1$ directly at the input terminals.

3.3 Use the solution of problem 3.2 to construct the response to a square-wave pulse of duration τ . (a) Relate the time constants of the line, the number of sections, and this pulse duration τ . (b) Discuss the contribution of the lowest order term and its relation to the signal shape.

3.4 Take two single sections of the R, C line, terminate at both ends into $R/2$, and find the output voltage applied at the sending end. Then insert between the two sections in series an inductance L and carry through again the solution for the applied square-wave pulse voltage. Compare the responses in both cases. Relate this to the idea of "loading."

3.5 Apply the impulse current $i_i(t) = M_i S_0(t)$ to the finite short-circuited cable in Fig. 3.2b. Find the output current.

3.6 Terminate the R - C line of N sections as shown in Fig. 3.2b at both ends into resistances $R/2$ and apply at the sending end the impulse voltage $v_0(t) = M_v S_0(t)$.

3.7 The finite high-pass filter of N sections with $Z = 1/(pC)$ and $Y = 1/(pL)$ has a step voltage $V_m I$ applied directly across the input terminals and is open-circuited at the far end. Find the output voltage.

3.8 The low-pass filter $Z = pL$, $Y = pC$ is terminated into the resistance $R = (L/C)^{1/2}$ at both ends. With a square-wave pulse voltage applied at the input end, find the output voltage.

3.9 Join in cascade two sections of a low-pass filter and two sections of a high-pass filter. Discuss the over-all steady-state characteristics if all L 's and all C 's have the same values.

3.10 In a low-pass filter of N sections, one section, say q , has unmatched values $L' = L + \Delta L$, $C' = C + \Delta C$. (a) Find the effect upon the filter response to a step voltage $V_m I$ if Δ signifies a small quantity. (b) Find the effect upon the steady-state propagation function.

3.11 The band-pass filter in Fig. 3.4 has a step voltage $V_m I$ applied at mid-series and is open-circuited at the far end. Find the output voltage $v_2(t)$.

3.12 In the filter in Fig. 3.3a, the series impedances are alternately pL_1 and pL_2 , whereas the shunt admittances are all alike and pC . Construe this as a uniform filter with unsymmetrical sections and (a) find the iterative impedances and propagation function; (b) evaluate the response to a step voltage $V_m I$ applied at mid-series with the line open-circuited.

3.13 In the filter in Fig. 3.3a, the series impedances are all alike and pL , whereas the shunt admittances are alternately pC_1 and pC_2 . Construe this as a uniform filter with unsymmetrical sections and (a) find the iterative impedances and propagation function; (b) evaluate the response to a step current $I_m I$ applied at mid-shunt with the line short-circuited.

3.14 Discuss the steady-state propagation functions for the two filters in problems 3.12 and 3.13 and compare them.

3.15 Take the lattice network of Fig. 3.5 with N like sections, replacing the inductances by resistances R . With a step voltage $V_m I$ applied, find the output voltage of the line over a terminal resistance R equal to the branch resistances.

3.16 Discuss the a-c steady-state characteristics of the lattice network in Fig. 3.5 but with inductances and capacitances interchanged. Compare with the discussion in the text.

3.17 In the R - C line of Fig. 3.2b let the number of sections go to infinity and apply at mid-series an impulse voltage $M_v S_0(t)$. Find the voltage and current anywhere along the line, i.e., for any value of q . Follow the general method in section 3.7.

3.18 In the problem of the moving train in Fig. 3.10a assume that the first mass is $M = 10$ m, but that all the k_α and f_α remain the same, as well as the f . Find the complete solution if an impulse force is applied at the first mass. In the analogue we apply an impulse voltage $M_v S_0(t)$ to the input terminals. Establish the criterion for nonoscillatory response.

3.19 The band-pass filter in Fig. 3.4 is composed of exactly N full shunt sections and $N + 1$ full series sections where $L_1 C_1 = 2L_2 C_2$. Find the output voltage across the last series branch if a step voltage $V_m I$ is applied in series with the first series branch.

3.20 Find the response of the infinite quasi-distortionless fourpole line to the impulse voltage $M_v S_0(t)$. The fourpole line in Fig. 3.10a is distortionless if we choose $R/L = G/C$. Assume the voltage applied at mid-series.

3.21 A quasi-distortionless fourpole line as in the previous problem is composed of N complete shunt branches and $N + 1$ complete series branches. Find the output voltage across the last series branch if a step voltage $V_m I$ is applied in series with the first series branch.

3.22 The quasi-distortionless line can be used for telemetering of signals. Apply to the finite line of the previous problem a square-wave voltage and compare the output voltage with it. Discuss the effect of the number of sections upon the signal shape.

3.23 Assume the infinite R - C line of Fig. 3.11a composed of sections of increasing resistance and decreasing capacitance, such that $R_n = (n + 1)R_0$, $C_n = C_0/(n + 1)$ if R_0 and C_0 are the values of the very first section. Find the response in the q 'th section to an impulse voltage $M_v S_0(t)$ applied at mid-series.

3.24 The process of anodic oxydation places layers of aluminum oxide on aluminum which possess different resistivities and exhibit polarization capacitances.

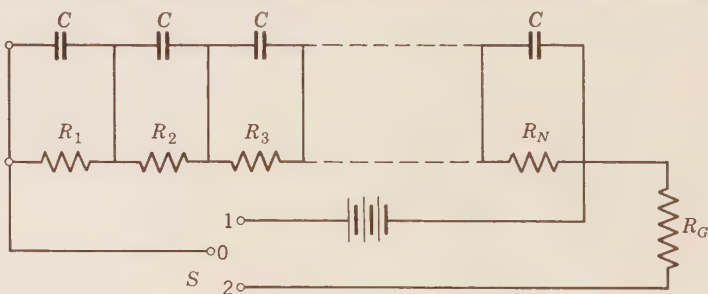


Fig. 3.13. Analogue to anodic oxydation layers.

Fig. 3.13 is an analogue with like capacitances C and different resistances R_α which might be assumed to increase with the order number α . (a) Describe the charging process when switch S closes 0-1 and places the voltage V across the network. (b) Describe the discharge process, when switch S closes 1-2 and places the galvanometer resistance R_G in series.*

* H. Bremner, "The Discharge of a Series of Equal Condensers, Having Arbitrary Resistances Connected in Parallel," *Philips Research Repts.*, **6**, 81-85 (1951).

4. IDEALIZED NETWORK CHARACTERISTICS (SYSTEM THEORY)

It was pointed out in the introduction to chapter 2 that communication systems are primarily cascade chains of fourpoles whereby each fourpole might in itself be a fairly complex network. The rigorous treatment in the foregoing two chapters of the transient performance of simple filter networks has amply demonstrated the involved nature of the mathematical solutions as well as the difficulties in interpreting the results in simple terms for practical applications. It has therefore become customary to idealize network characteristics, even at the expense of losing accuracy in detail, in order to gain greater insight into systems characteristics. Moreover, the recent sharpening of requirements for greater fidelity in the intelligence received in television, follow-up systems, pulse coding systems, etc., has placed greater and greater emphasis upon approaching idealized characteristics for the system components so that a brief treatment of this aspect appears indicated. We shall employ here primarily the Fourier transform method because it has been found more directly suitable to describe the systems functions as functions of real frequency.

4.1 Idealization of Network Characteristics

To formulate the general case of the transmission of a signal through a network, it is convenient to choose a fourpole as in Fig. 4.1, terminated in like pure resistances at both terminal pairs, and supply the input signal either from a current source $i_0(t)$ or from a voltage source $v_0(t)$. The use of resistances as terminal devices is arbitrary but often preferred. In section 2.3 we deduced the relations in terms of Laplace transforms, obtaining (2.36) or better

$$I_2(p) = [\mathcal{C}R^2 + (\mathcal{A} + \mathcal{D})R + \mathcal{B}]^{-1} \begin{cases} V_0(p) \\ RI_0(p) \end{cases} \quad (1)$$

where \mathcal{A} , \mathcal{B} , \mathcal{C} , \mathcal{D} are the general fourpole parameters. We can define various transfer characteristics, such as

$$\frac{RI_2(p)}{V_0(p)} = H(p), \text{ transfer function}$$

$$\frac{I_2(p)}{V_0(p)} = Y_T(p), \text{ transfer admittance}$$

$$\frac{V_0(p)}{I_2(p)} = Z_T(p), \text{ transfer impedance}$$

For our further discussion we will select $H(p)$, the transfer function, which becomes from (1)

$$H(p) = \frac{R}{\mathfrak{A} + (\mathfrak{A} + \mathfrak{D})R + \mathfrak{C}R^2} \quad (2)$$

For any given network we can readily obtain the fourpole parameters by the processes outlined at the end of section 2. Replacing p , the

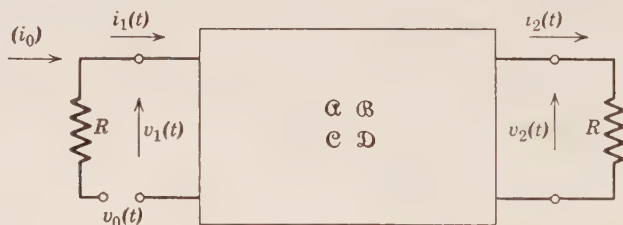


Fig. 4.1. General fourpole with balanced resistive terminations.

complex frequency, by $j\omega$ or, what leads to the same result, letting $\delta \rightarrow 0$ in $p = \delta + j\omega$, we obtain the transfer function in complex form

$$\begin{aligned} H(j\omega) &= |H(j\omega)|e^{-j\Phi(\omega)} \\ &= H_1(\omega) - jH_2(\omega) \end{aligned} \quad (3)$$

Assuming that $H(j\omega)$ is an integer function of $j\omega$, either rational or meromorphic, then separating real and imaginary parts, $H_1(\omega)$ will be an even function and $H_2(\omega)$ an odd function of ω . Because

$$\begin{aligned} |H(j\omega)| &= [H_1^2(\omega) + H_2^2(\omega)]^{1/2} = h(\omega) \\ \Phi(\omega) &= \tan^{-1} \frac{H_2(\omega)}{H_1(\omega)} \end{aligned} \quad (4)$$

the amplitude function $h(\omega)$ is an even function of ω and the phase function $\Phi(\omega)$ an odd function.

Let us introduce the schematic diagram of Fig. 4.2 to indicate the action of a fourpole with transfer function $H(j\omega)$ upon the signal

$v_s(t)$ applied at the input terminal pair. We designate the “received” voltage at the output terminal pair by $v_r(t)$. Defining the Fourier

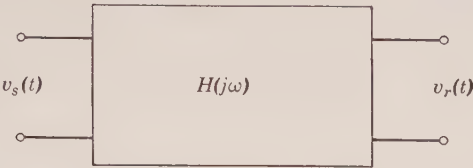


Fig. 4.2. Schematic of transfer action of fourpole.

transforms of the input signal in accordance with (1.30)

$$\mathfrak{F}v_s(t) = \bar{V}_s(\omega) = \int_{-\infty}^{+\infty} v_s(t)e^{-j\omega t} dt \tag{5}$$

and that of the received signal voltage by

$$\mathfrak{F}v_r(t) = \bar{V}_r(\omega) = \int_{-\infty}^{+\infty} v_r(t)e^{-j\omega t} dt \tag{6}$$

then the following relation exists between these Fourier transforms

$$\bar{V}_r(\omega) = H(j\omega)\bar{V}_s(\omega) \tag{7}$$

quite similar to (1.36). For a given signal $v_s(t)$ the complex transform

$$\bar{V}_s(\omega) = V_1(\omega) + jV_2(\omega) = |\bar{V}_s(\omega)|e^{j\psi_s(\omega)} \tag{8}$$

can readily be evaluated. In fact, Figs. 1.2 through 1.9 illustrate the spectrum functions for two distinct pulse types in several equivalent presentations. With $\bar{V}_s(\omega)$ found, the received signal can then be obtained as the inverse Fourier transform of (7), namely

$$v_r(t) = \mathfrak{F}^{-1}\bar{V}_r(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(j\omega)\bar{V}_s(\omega)e^{j\omega t} d\omega \tag{9}$$

It is now obvious that identity of $v_r(t)$ with $v_s(t)$, or *ideal transmission*, requires

$$\begin{aligned} H(j\omega) &= 1 \\ h(\omega) &= 1, \quad \Phi(\omega) = 0 \end{aligned}$$

However, this requirement could be fulfilled only with an ideal resistance network.

Another important special case arises if we let the amplitude be constant and the phase function be proportional to ω , namely

$$h(\omega) = A, \quad \Phi(\omega) = \omega t_d, \quad -\infty < \omega < \infty \tag{10}$$

This leads in (9) to

$$v_r(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A e^{-j\omega t_d} \bar{V}_s(\omega) e^{j\omega t} d\omega$$

From Table 1.2, line 2, we take the exponential factor to indicate a time shift, so that

$$v_r(t) = A v_s(t - t_d) \quad (11)$$

The signal is exactly reproduced with a scale factor A and with a time delay t_d . We call this *distortionless transmission*. Unfortunately, as we have stressed in section 1.2, all the spectrum functions extend over the entire infinite range of frequencies so that $h(\omega)$ would need to be constant and $\Phi(\omega)$ would need to be linear over the same infinite range in order to guarantee distortionless reproduction of the signal.

Any deviation from conditions (10) will thus lead to distortion. Depending upon the specific type of deviation, we differentiate between *amplitude-frequency distortion* resulting from the variation of $h(\omega)$ with frequency, and *phase-frequency distortion* resulting from a lack of proportionality of phase shift to frequency. Both effects are essentially linear effects because the spectrum concept is tied to linear circuit relationships. In general, nonlinear effects might also occur which lead to different types of distortion, though the terminology has been somewhat confused.*

Actually, no network has "infinite" bandwidth, i.e., can satisfy conditions (10) for all values of ω . One can rephrase this by saying that every network acts as a filter and the proper system design will select all system components with nearly identical frequency characteristics in order to insure the desired over-all system characteristics. One can differentiate, broadly speaking,

1. Low-pass systems.
2. Narrow band-pass systems (highly selective).
3. Wide-band systems.

The last category includes the so-called high-pass systems which, strictly speaking, exist only within arbitrarily restricted frequency regions, such as audio, video, and the like.

For distortionless transmission of *steady-state periodic signals*, it is rather simple to define *ideal filters* fitting the specified systems characteristics, namely:

$$h(\omega) = A, \quad \Phi(\omega) = \omega t_d, \quad \omega_1 < \omega < \omega_2 \quad (12)$$

* An attempt to disentangle conflicting uses of the same terms was made in the I.R.E. Standard, "Definitions of Terms in the Field of Linear Varying Parameter and Nonlinear Circuits," *Proc. I.R.E.*, **42**, 554-555 (1954).

i.e., constant attenuation and linear phase shift over the specified frequency range and total suppression outside this frequency range.* Again, any periodic signal $s(t)$ has theoretically harmonic components up to any order, as the Fourier series

$$s(t) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\omega_0 t + B_n \sin n\omega_0 t) \quad (13)$$

clearly indicates. However, to have meaning, this series must be convergent so that for all practical purposes the upper limit will be a finite number N defined by the desired degree of fidelity. Choosing then the

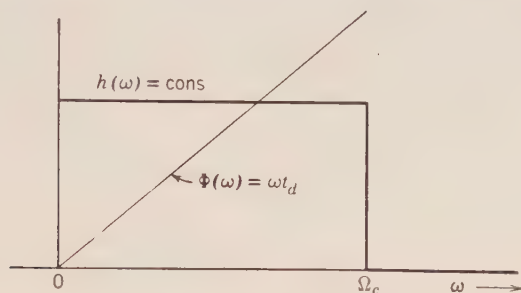


Fig. 4.3. Ideal low-pass filter characteristics. Amplitude $h(\omega)$ is an even function, and phase $\Phi(\omega)$ an odd function of frequency.

ideal characteristic of Fig. 4.3 with $\Omega_c = N\omega_0$ where ω_0 is the fundamental periodicity of the signal, or *repetition frequency*, we can be assured of “distortionless” transmission in a practical sense. If $\Omega_c < N\omega_0$, then we clearly have signal distortion in the senses defined in the foregoing. Actually, in Fig. 4.3 we have not needed to specify at all the phase characteristic beyond the cutoff frequency Ω_c because we have assumed zero amplitude for $\omega > \Omega_c$.

In real filters, this sharp cutoff characteristic cannot be achieved. For example, in Fig. 3.6 we have pictured the frequency characteristics of the infinite simple low-pass filter in accordance with the analytic expressions (3.74). To show the entire characteristic, the abscissae are chosen as ω/Ω_c for $\omega < \Omega_c$ and as Ω_c/ω for $\omega > \Omega_c$. We see that with an increasing number of sections the amplitude characteristic $h(\omega)$ will approach the ideal case; in practice, economy would dictate the maximum number of sections compatible with the desired results.

* H. W. Bode and R. C. Dietzold, “Ideal Wave Filters,” particularly in introduction, *Bell System Tech. J.*, **14**, 215–252 (1935).

We observe, though, that the phase shift might become an important characteristic because we have a definite constant delay.

Taking then only two sections symmetrically terminated into resistances as shown in Fig. 4.4, we can use (3.39) with the fourpole parameters from (3.47) for the cascade of the two sections. We also need to

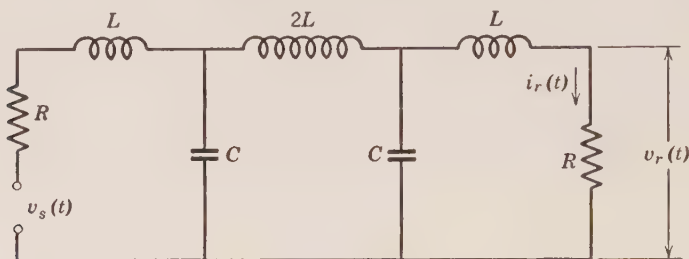


Fig. 4.4. Cascade of two simple low-pass filter sections with symmetrical resistive terminations.

replace p by $j\omega$ in order to get Fourier transform relations rather than Laplace transforms, so that we have

$$\bar{V}_r(\omega) = R\bar{I}_r(\omega) = \frac{2\alpha c R}{2(2\alpha c Z_t + 2\alpha^2)(2\alpha c Z_t + 2\alpha^2 - 2)} \bar{V}_s(\omega)$$

For the T-sections we also have

$$\alpha = 1 + YZ, \quad c = Y$$

and finally

$$Y = j\omega C, \quad Z = j\omega L, \quad Z_t = R$$

Thus we arrive after obvious simplifications by cancellations of terms at

$$\begin{aligned} H(j\omega) &= \frac{\bar{V}_r(\omega)}{\bar{V}_s(\omega)} \\ &= \frac{1}{2} (1 - \omega^2 LC + j\omega CR)^{-1} \left[1 - \omega^2 LC + j\omega \frac{2L}{R} \left(1 - \frac{\omega^2}{2} LC \right) \right]^{-1} \end{aligned}$$

Furthermore we define as in section 2.3

$$\Omega_c^2 = \frac{2}{LC}, \quad \frac{\omega}{\Omega_c} = x, \quad Q_c = \frac{\Omega_c L}{R}$$

so that

$$\frac{1}{2H(j\omega)} = \left(1 - 2y^2 + jy \sqrt{\frac{2}{Q_c}} \right) [1 - 2y^2 + jy^2 \sqrt{2Q_c} (1 - y^2)]$$

To plot a simple case, we choose $Q_c = 2$, and thus obtain finally

$$\begin{aligned}\frac{1}{2H(j\omega)} &= (1 - 8y^2 + 8y^4) + jy(5 - 14y^2 + 8y^4) \\ &= U(y) + jW(y)\end{aligned}$$

so that

$$2h(\omega) = (U^2 + W^2)^{-1/2}, \quad \Phi(\omega) = \tan^{-1} \frac{W}{U}$$

Figure 4.5 gives the graphical representation of both amplitude and phase functions, showing even in this case reasonable similarity to the

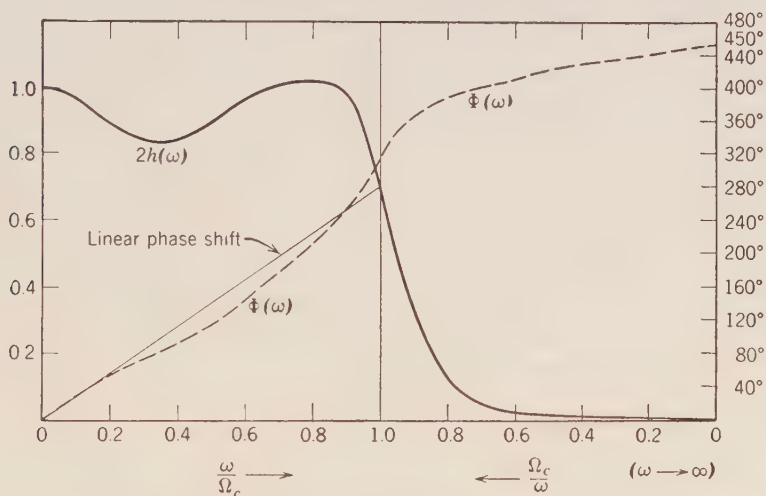


Fig. 4.5. Amplitude and phase characteristics of network shown in Fig. 4.4. $h(\omega)$ is an even function and $\Phi(\omega)$ is an odd function of frequency.

ideal low-pass characteristics of Fig. 4.3. Again we observe that the most serious omission in the ideal case appears to be the complete disregard of the phase function outside the pass band. We might also note here that the most significant deviations within the pass band are the oscillations of both amplitude and phase functions about arbitrarily interpolated straight-line characteristics. Similar graphs are shown in Cherry,^{A4} pp. 88, 128, 141, and 180, for various simple but typical filter circuits. Experimental confirmation has been given with a specially designed curve tracer showing both amplitude and phase characteristics.*

* B. D. Loughlin, "A Phase Curve Tracer for Television," *Proc. I.R.E.*, **29**, 107-115 (1941).

We conclude that the idealization of filter characteristics by substitution of constant attenuation and linear phase shift over the desired frequency region will be satisfactory for steady-state periodic signals if all the significant harmonic components can be included in this frequency range. Should a significant part of the signal line spectrum fall beyond the selected filter frequency range, distortion must be expected both because of the decrease of the attenuation characteristic and the strong deviation of the phase characteristic from proportionality with frequency.

4.2 Response of Ideal Filters to Pulses

If we consider signals made up of single pulses or pulse groups, we must realize from the previous section that any finite pass band of frequencies will generally be insufficient to reproduce the pulse form with accuracy. In Figs. 1.2 through 1.9 we plotted the spectrum functions for two series of pulses. Though it is correct that the most significant part of the spectrum appears to lie within the frequency range approximately equal to the inverse of the pulse duration, it would lead to disappointment to accept this with equal confidence as in the steady-state periodic signal. The idealization of the network characteristics will have to be considered more carefully. The most extensive systematic study of network performance from this systems point of view on the basis of idealized network characteristics was done by Küpfmüller,^{A10} who was also the first to formulate the transient problem in idealized filters and its characteristic aspects.*

Suppose we take the ideal low-pass filter with the characteristics in Fig. 4.3 and apply to it a step voltage $V_0 1$. The Fourier transform of the step voltage is from Table 1.1, line 1, with $\alpha \rightarrow 0$,

$$\bar{V}_s(\omega) = \frac{V_0}{j\omega}$$

We find the response from (9) if we use the ideal filter characteristics (12) and adjust the limits in accordance with Fig. 4.3

$$v_r(t) = \frac{1}{2\pi} \int_{-\Omega_0}^{+\Omega_0} A e^{-j\omega t_d} \frac{V_0}{j\omega} e^{j\omega t} d\omega \quad (14)$$

We could consider this an integral along the real frequency axis of ω , but we would need to be careful of the point $\omega = 0$ where the integrand

* K. Küpfmüller, "Transients in Wave Filters" [German], *Elek. Nachr. Tech.* **1**, 141-152 (1924); also "Relations Between Frequency Characteristics and Transient Performance in Linear Systems" [German], *Ibid.*, **5**, 18-32 (1928). See also Guillemin,^{A8} Vol. II, pp. 477-494.

takes on infinite values and the integral becomes improper as shown in Guillemin,^{A8} Vol. II, p. 472. The existence of the pole at $\omega = 0$ suggests, however, an evaluation in the complex plane of $p = \delta + j\omega$ because it lets us utilize the residue theorem.

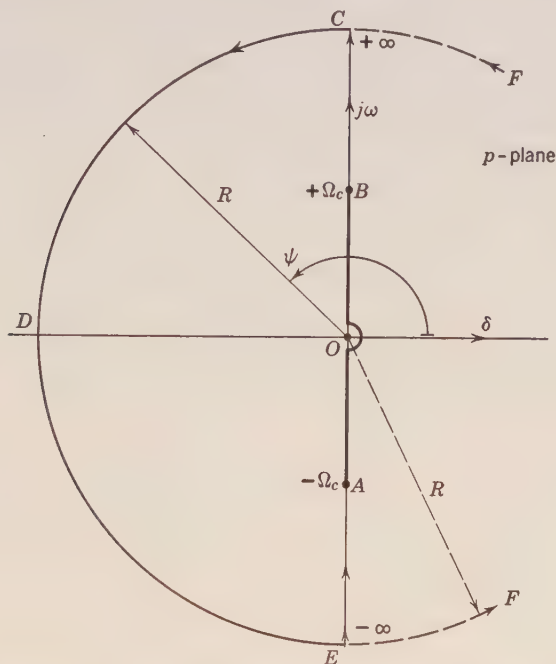


Fig. 4.6. Integration in the complex p -plane to evaluate indicial response of ideal low-pass filter.

We first change the variable in (14) to $j\omega$, so that

$$v_r(t) = \frac{AV_0}{2\pi j} \int_{-j\Omega_c}^{+j\Omega_c} \frac{1}{j\omega} e^{j\omega(t-t_d)} d(j\omega) \quad (15)$$

In Fig. 4.6, depicting the conventional p -plane, the path of integration lies along the imaginary axis skirting around the pole $p = 0$ as we normally have done in Laplace transform evaluation; see Fig. 1.11. The integral (15) can now be taken as a part of the closed line integral utilizing the nearly infinite semicircle of radius $R \rightarrow \infty$. Thus

$$\int_{-j\Omega_c}^{+j\Omega_c} = \oint_{EABCDE} - \int_{j\Omega_c}^{j\infty} - \int_{CDE} - \int_{-\infty}^{-j\Omega_c} \quad (16)$$

We shall now evaluate each contribution in turn with the abbreviation $t - t_d = \tau$. The closed line integral is by the residue theorem

$$\mathcal{F} = 2\pi j(e^{j\omega\tau})_{\omega=0} = 2\pi j \quad (17)$$

The next integral can be separated into two real variable integrals

$$\int_{j\Omega_c}^{j\infty} \frac{1}{j\omega} e^{j\omega\tau} d(j\omega) = \int_{\Omega_c}^{\infty} \frac{\cos \omega\tau}{\omega} d\omega + j \int_{\Omega_c}^{\infty} \frac{\sin \omega\tau}{\omega} d\omega \quad (18)$$

Both integrals are well-known functions, tables of which are given for example in Jahnke-Emde.* The conventional definitions are so arranged that the integrals remain analytic functions. Thus

$$\begin{aligned} Si(x) &= \int_0^x \frac{\sin u}{u} du \\ Ci(x) &= - \int_x^{\infty} \frac{\cos u}{u} du \end{aligned} \quad (19)$$

respectively known as sine integral and cosine integral and shown graphically in Fig. 4.7. We observe that $Ci(x)$ cannot be extended

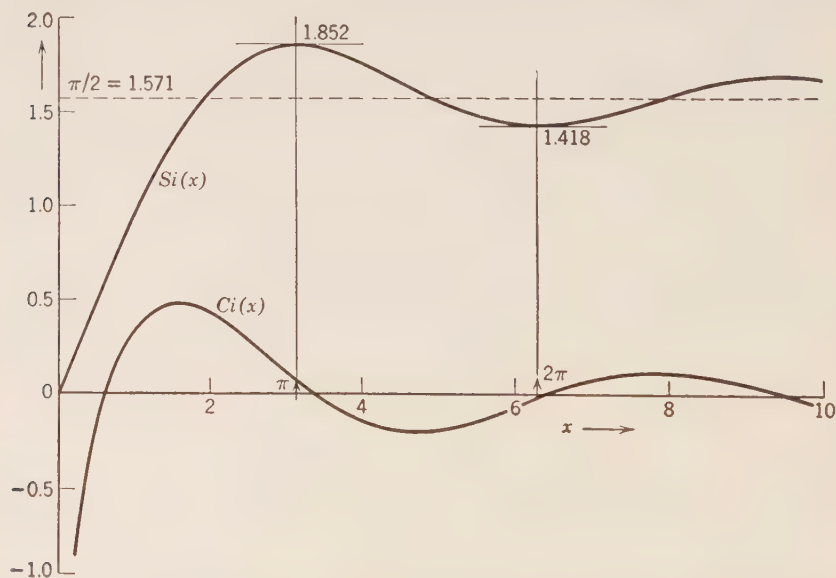


Fig. 4.7. The functions sine integral $Si(x)$ and cosine integral $Ci(x)$ defined by (19).

in its lower limit to $x = 0$ because of the singularity at $x = 0$. Actually, this is the reason why (15) cannot be evaluated in the simple

* E. Jahnke and F. Emde, *Tables of Functions*, 3rd edition, pp. 3-9, Teubner, Leipzig, 1938; also Dover Publications, New York.

form in which it appears. Now we can write

$$\begin{aligned} \int_{\Omega_c}^{\infty} \frac{\sin \omega \tau}{\omega} d\omega &= \int_0^{\infty} \frac{\sin \omega \tau}{\omega \tau} d(\omega \tau) - \int_0^{\Omega_c \tau} \frac{\sin \omega \tau}{\omega \tau} d(\omega \tau) \\ &= \frac{\pi}{2} - Si(\Omega_c \tau) \end{aligned}$$

The total integral (18) is therefore

$$\int_{j\Omega_c}^{j\infty} \frac{1}{j\omega} e^{j\omega\tau} d(j\omega) = -Ci(\Omega_c\tau) + j\frac{\pi}{2} - jSi(\Omega_c\tau) \quad (20)$$

The integral over the infinite semicircle can be taken in exactly the same manner as demonstrated for the inverse Laplace transform of unit step. We use instead of $j\omega$ the complex frequency $p = \delta + j\omega$ and along the path of integration we have

$$\begin{aligned} p &= Re^{j\psi} = R \cos \psi + jR \sin \psi \\ dp &= jRe^{j\psi} d\psi \end{aligned}$$

This brings the integral into the form

$$\int_C^E e^{p\tau} \frac{dp}{p} = j \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \exp(R\tau \cos \psi + jR\tau \sin \psi) d\psi \rightarrow 0$$

Because $\cos \psi \leq 0$ for the values of ψ in the integration interval, the contribution vanishes for $\tau > 0$ as $R \rightarrow \infty$, where one needs to be a little careful near the limits as outlined in Vol. I, pp. 207–208. The last integral can be treated like (18) and readily leads to

$$\int_{-j\infty}^{-j\Omega_c} \frac{1}{j\omega} e^{j\omega\tau} d(j\omega) = \int_{\infty}^{\Omega_c} \frac{1}{\omega} e^{-j\omega\tau} d\omega = +Ci(\Omega_c\tau) + j\frac{\pi}{2} - jSi(\Omega_c\tau) \quad (21)$$

The desired integral (15) becomes after collection of all the terms as indicated in (16),

$$\begin{aligned} v_r(t) &= \frac{AV_0}{2\pi j} \left(2\pi j + Ci - j\frac{\pi}{2} + jSi - Ci - j\frac{\pi}{2} + jSi \right) \\ &= AV_0 \left[\frac{1}{2} + \frac{1}{\pi} Si[\Omega_c(t - t_d)] \right] \quad (22) \end{aligned}$$

Actually, this result (22) is valid also for $\tau < 0$. As demonstrated for the general Laplace transform, the path of integration for $\tau < 0$ must be closed over the infinite semicircle in the right half plane of

Fig. 4.6 in order to get a convergent and thus meaningful result. If we do this here, we need to reverse the direction of integration to observe the rule that the region bounded by the line integral lies to the left of it. We have then, instead of (16)

$$\int_{+j\Omega_c}^{-j\Omega_c} = \oint_{CBAEFC} - \int_{-j\Omega_c}^{-j\infty} - \int_{EFC} - \int_{+j\infty}^{+j\Omega_c} \quad (23)$$

The contribution of the first, closed integral in the right half plane vanishes by Cauchy's theorem since no pole is enclosed. The third integral on the right-hand side also gives zero contribution as $R \rightarrow \infty$, because $\tau < 0$, keeping in mind the needed care near the limits pointed out in the foregoing. The remaining two integrals are treated identically to (20) and (21) and give the same results with one exception. We have here, because of $\tau < 0$

$$\int_0^\infty \frac{\sin \omega \tau}{\omega \tau} d(\omega \tau) = - \int_0^\infty \frac{\sin \omega |\tau|}{\omega |\tau|} d(\omega |\tau|) = - \frac{\pi}{2}$$

so that the terms $j(\pi/2)$ in (20) and (21) would reverse their signs. This gives then for (15) with the contributions as in (23) and with attention to the orders of the limits

$$\begin{aligned} v_r(t) &= - \frac{AV_0}{2\pi j} \left(-Ci - j\frac{\pi}{2} - jSi + Ci - j\frac{\pi}{2} - jSi \right) \\ &= AV_0 \left[\frac{1}{2} + \frac{1}{\pi} Si[\Omega_c(t - t_d)] \right] \end{aligned} \quad (24)$$

confirming (22).

The final result is plotted in Fig. 4.8 as the indicial response in the normalized form

$$\frac{v_r(t)}{AV_0} = \mathcal{R}(x) = \frac{1}{2} + \frac{1}{\pi} Si(x) \quad (25)$$

where $x = \Omega_c(t - t_d)$. We recognize at once the, on first impression, strange phenomenon that the response is odd symmetrical about $x = 0$, where $\mathcal{R}(x) = \frac{1}{2}$, so that it extends infinitely far into negative values of x , or in time far beyond

$$t = 0, \quad x_0 = -\Omega_c t_d$$

when the step voltage $V_0 I$ was applied to the input terminals. This is obviously a physical impossibility but can be explained readily by the fact that we have arbitrarily assumed ideal amplitude *and* phase characteristics. Since the impedance functions of realizable linear

networks must be analytic functions of the complex frequency p , we probably have been *too arbitrary* and need to retreat somewhat.

However, if we tolerate for this discussion the nonphysical response characteristic and examine the information conveyed by (25) we see that we might find ample compensation in the elegant and extremely basic interpretation of the result. As Fig. 4.8 indicates, the abrupt input step shown by dash lines is transformed into a sloping front and the slope is directly and only dependent upon the cutoff frequency. We might define a *build-up time* τ_B in terms of Δx , the distance between

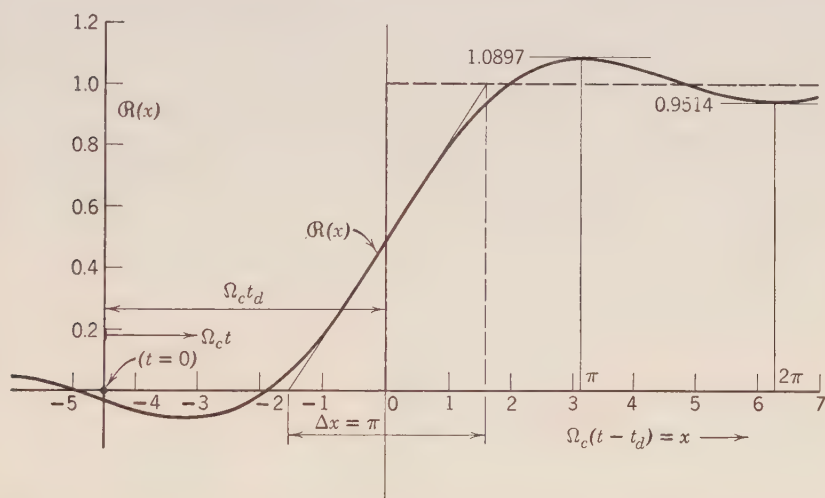


Fig. 4.8. Response of ideal low-pass filter to unit step. The time origin is at $\Omega_c t_d = x_0$ to the left of $x = 0$.

the intersections of the straight line tangent to the response curve at $x = 0$ which is its center of symmetry. The slope of $R(x)$ is from (25) with the definition (19)

$$\left(\frac{d}{dx} R(x) \right)_{x=0} = \frac{1}{\pi} \left(\frac{\sin x}{x} \right)_{x=0} = \frac{1}{\pi}$$

From Fig. 4.8 we take then $\Delta x = \pi$, but this must also be $\Omega_c \tau_B$, so that we obtain

$$\tau_B = \frac{\pi}{\Omega_c} = \frac{1}{2f_c} \quad (26)$$

where f_c is the cutoff frequency and $2f_c$ the total width (positive and negative) of the continuous spectrum amplitude characteristic $h(\omega)$

from Fig. 4.3. The larger f_c , the quicker the build-up, and for $f_c \rightarrow \infty$ we approach distortionless transmission as is to be expected.

The delay t_d of the response is directly defined by the slope of the phase characteristic $\Phi(\omega)$. If we use $x_0 = -\Omega_c t_d$, then, referred to the time origin which is the start of the applied signal voltage, we have in the conventional form

$$\Omega_c t = x - x_0$$

Since the phase shift for a single section of the low-pass filter in Fig. 3.6 is π , we see that for it x_0 will be of the order of π and thus τ_B of the same order as t_d . As the number of sections increases, the slope of the phase characteristic in Fig. 4.3 must of course also increase and thus the delay t_d will be roughly proportional to the number of sections of the filter, whereas the build-up time remains the same as in (26).

The shape of the response indicates a normalized overshoot of 8.97% independent of the width of the transmission frequency band. The specific wave shape depends, however, upon both the signal and the assumed network characteristic. It is therefore not possible to establish a single general criterion for signal distortion.

Suppose we consider now a rectangular pulse of duration T which can be expressed

$$v_s(t) = V_0 S_{-1}(t) - V_0 S_{-1}(t - T) \quad (27)$$

as the difference of two step voltages $V_0 I$ spaced T apart. We use here the notation of Campbell-Foster^{E1} for the unit step because it is simpler to designate the time shift. The response of the ideal low-pass filter of Fig. 4.3 can be obtained in the same manner as the superposition of (25) and its delayed version, namely

$$\begin{aligned} \frac{v_r(t)}{A V_0} &= \frac{1}{2} + \frac{1}{\pi} Si(x) - \left(\frac{1}{2} + \frac{1}{\pi} Si(x - x_T) \right) \\ &= \frac{1}{\pi} [Si(x) - Si(x - x_T)] \quad (28) \end{aligned}$$

where $x_T = \Omega_c T$, the normalized pulse duration. We can here cancel the terms $\frac{1}{2}$ because each of the expressions is valid for $-\infty < t < +\infty$ (note the difference between this and the one-sided Laplace transform, Vol. I, section 5.5). As long as the pulse duration is larger than the build-up time, $T > \tau_B$, the pulse will reach its maximum amplitude and remain at it for the interval $T - \tau_B$. If, however, $T < \tau_B$, then the pulse amplitude will become a measure of the pulse duration as is well known in long-distance telegraphy.*

* J. R. Carson, *Electric Circuit Theory and the Operational Calculus*, McGraw-Hill, New York, 1929.

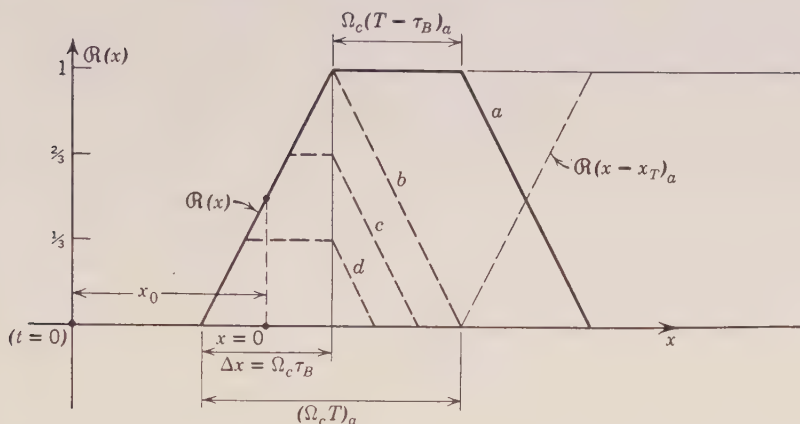


Fig. 4.9. Idealized output pulse shapes as function of the pulse duration T relative to the build-up time τ_B . (a) $T = 2\tau_B$; (b) $T = \tau_B$; (c) $T = \frac{2}{3}\tau_B$; (d) $T = \frac{1}{3}\tau_B$.

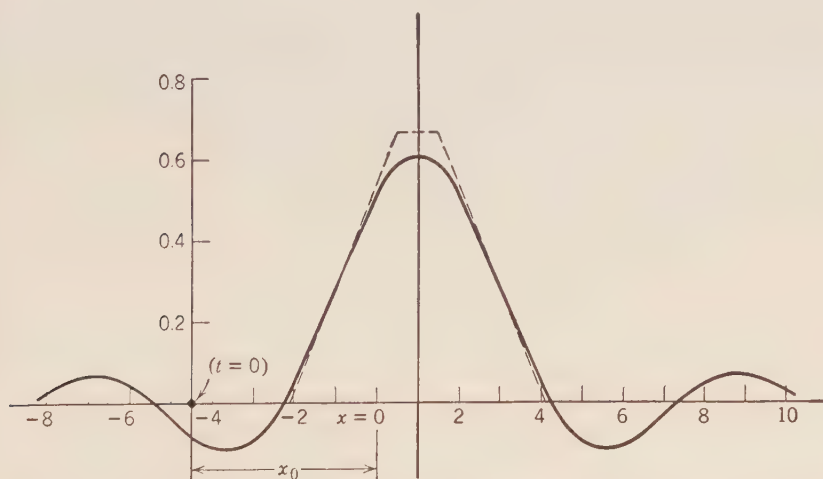


Fig. 4.10. Actual pulse shape from (28) corresponding to the idealized pulse shape c of Fig. 4.9.

To obtain the simplest illustration, we have plotted in Fig. 4.9 the response $\mathcal{R}(x)$ in idealized straight lines, leaving off the oscillations so apparent in Fig. 4.8. For $x_T = 2\Delta x = 2\Omega_c\tau_B$ pulse shape a shows that the maximum amplitude is held for the interval $\Omega_c(T - \tau_B)$ and the pulse is symmetrical. Pulse shape b corresponds to the limiting case $T = \tau_B$, and shapes c and d exhibit the proportionality of the amplitude to the pulse duration. The actual pulse shape has been

plotted in Fig. 4.10 for the idealized pulse shape c of Fig. 4.9 which is also superimposed in dash lines.

If we let the pulse duration become extremely small and at the same time let V_0 , the amplitude, increase beyond all limits, we approach the impulse function. We can derive the response from (28), by observing

$$\lim_{x_T \rightarrow 0} [Si(x) - Si(x - x_T)] \rightarrow \frac{\sin x}{x} x_T$$

either by a Taylor series expansion of $Si(x - x_T)$ or by considering the difference to become the differential of $Si(x)$, which from (19) is the above with $x_T \rightarrow 0$. We obtain therefore as response to the impulse function $V_0 T S_0(t)$, where $S_0(t)$ is the unit impulse (1.47) or (1.48)

$$\frac{v_r(t)}{A} = \frac{V_0 T}{\pi} \Omega_c \frac{\sin x}{x} \quad (29)$$

where again $x = \Omega_c(t - t_d)$. The same result is obtained if we return to the original Fourier integral (14) and replace the Fourier transform of unit step by that of the unit impulse given in (1.48), namely

$$v_r(t) = \frac{1}{2\pi} \int_{-\Omega_c}^{+\Omega_c} A e^{-j\omega t_d} V_0 T e^{j\omega t} d\omega \quad (30)$$

The integration is readily performed and leads to

$$\int_{-\Omega_c}^{+\Omega_c} e^{j\omega(t-t_d)} d\omega = \frac{1}{j(t-t_d)} 2j \sin \Omega_c(t-t_d)$$

Collecting the terms and introducing x as before, we arrive at (29). Actually, the plot of this impulse response is very near the same as Fig. 4.10 except that the amplitude is directly proportional to the cut-off frequency! If we define the base of the impulse response τ_i by the distance of the first zeros to the right and left of the center, then

$$\frac{\Omega_c \tau_i}{2} = \pi, \quad \tau_i = \frac{2\pi}{\Omega_c} = \frac{1}{f_c} \quad (31)$$

This shows that the impulse response deteriorates as the cutoff frequency is lowered, the pulse response becomes broader at the base, and the amplitude decreases.

These few examples have amply demonstrated the power of this concept of idealized network functions, introduced by Küpfmüller^{A10} and illustrated in much detail in this reference. It should be obvious that in a sequence of like or unlike pulses the spacing will be an addi-

tional parameter of interest; permissible values will be closely related to the build-up time and thus to the transmission bandwidth.

4.3 A-C Response of Ideal Filters

Again we shall consider first the ideal low-pass filter and apply a single frequency carrier wave, say

$$v_s(t) = V_m \sin(\omega_0 t + \psi) = \text{Im } \bar{V} e^{j\omega_0 t} \quad (32)$$

The Fourier transform in complex form is from Table 1.1, line 1, with $\alpha = j\omega_0$

$$\bar{V}_s(\omega) = \frac{\bar{V}}{j(\omega - \omega_0)}$$

which needs to be substituted in (14) for the transform of unit step. We obtain in complex notation

$$\bar{v}_r(t) = \frac{1}{2\pi} \int_{-\Omega_c}^{+\Omega_c} A e^{-j\omega t_d} \frac{\bar{V}}{j(\omega - \omega_0)} e^{j\omega t} d\omega \quad (33)$$

Though a seemingly very slight change, the effect of the extra ω_0 is quite important. We can bring (33) into the same form as the integral (15) if we introduce the new variable

$$y = \omega - \omega_0, \quad dy = d\omega$$

and adjust the limits accordingly

$$\begin{aligned} \omega = -\Omega_c, & \quad y = -(\Omega_c + \omega_0) \\ \omega = +\Omega_c, & \quad y = +(\Omega_c - \omega_0) \end{aligned}$$

This leads with $t - t_d = \tau$ to

$$\bar{v}_r(t) = \frac{A\bar{V}}{2\pi j} e^{j\omega_0 \tau} \int_{-j(\Omega_c + \omega_0)}^{+j(\Omega_c - \omega_0)} \frac{1}{jy} e^{jy\tau} d(jy) \quad (34)$$

The integrand is now identical with that in (15), but the limits are unsymmetrical, being shifted along the imaginary axis by $-j\omega_0$. Comparing with the path of integration in Fig. 4.6 we observe this simple shift quite strikingly. Assuming $\Omega_c > \omega_0$, i.e., the carrier within the transmission band of the filter, nothing changes in principle from our previous procedure and, indeed, we can utilize all the results from the d-c case with appropriate slight modifications. We can again use (16), except for the finite limit values which are now different. This means that only (20) and (21) will change in argument and we can readily write the expression corresponding to (22), here in

complex notation, namely

$$\bar{v}_r(t) = \frac{A \bar{V}}{2\pi j} e^{j\omega_0 \tau} \left(2\pi j + Ci(\Omega_c - \omega_0)\tau - j\frac{\pi}{2} + jSi(\Omega_c - \omega_0)\tau \right. \\ \left. - Ci(\Omega_c + \omega_0)\tau - j\frac{\pi}{2} + jSi(\Omega_c + \omega_0)\tau \right) \quad (35)$$

Because of the unsymmetry of the limits, the cosine integral terms do not cancel, and the sine integral terms cannot be combined. We actually have

$$\bar{v}_r(t) = A \bar{V} e^{j\omega_0(t-t_d)} \left(\frac{1}{2} + \frac{1}{2\pi} (Siy' + Siy'') + \frac{j}{2\pi} (Ciy'' - Ciy') \right) \quad (36)$$

where

$$y' = (\Omega_c - \omega_0)(t - t_d), \quad y'' = (\Omega_c + \omega_0)(t - t_d)$$

Comparing (36) with the step response (22), we see that within the large parentheses the first two terms are quite similar in both cases, and if we let $\omega_0 \rightarrow 0$ we obtain exactly the large parentheses in (22). The step voltage amplitude V_0 in (22) corresponds here to the applied alternating voltage $\bar{V}e^{j\omega_0 t}$ in complex notation and with the same time delay that characterizes the indicial response. As a matter of fact, we can readily interpret the terms within the large parentheses as the *envelope* of the carrier frequency wave, giving amplitude and phase variation for the transient state. Introducing

$$g_1(t) = \frac{1}{2} + \frac{1}{2\pi} (Siy'' + Siy') \\ g_2(t) = \frac{1}{2\pi} (Ciy'' - Ciy') \quad (37)$$

and also

$$\tan \eta = \frac{g_2(t)}{g_1(t)}$$

we can write (36) in accordance with (2.100)

$$v_r(t) = A V_m \sqrt{g_1^2(t) + g_2^2(t)} \sin [\omega_0(t - t_d) + \psi + \eta] \quad (38)$$

This clearly exhibits the amplitude variation which is also shown in Fig. 4.11 for the special case $\omega_0 = \frac{1}{2}\Omega_c$. Again we might compare this with the response to a step voltage shown in Fig. 4.8. We observe a slightly longer build-up time for the a-c case and a definite lack of symmetry with respect to the ordinate at $x = 0$. The closer ω_0 lies to Ω_c , the stronger will be this unsymmetry.

In plotting Fig. 4.11 we had to compute $g_2(t)$ from (37) for very small values of x where the cosine integral increases beyond all limits. We can take from Jahnke-Emde, *op. cit.*, p. 3, the approximation

$$Ci(x) \approx \ln \gamma x \quad x \ll 1 \quad (39)$$

where $\ln \gamma = 0.5772$ is Euler's constant. Since we have, for $\omega_0 = \frac{1}{2}\Omega_c$, $y' = \frac{1}{2}x$ and $y'' = \frac{3}{2}x$, we can approximate

$$g_2(t) = \frac{1}{2\pi} \left(Ci \frac{3}{2}x - Ci \frac{1}{2}x \right) \approx \frac{1}{2\pi} \ln 3 \quad x \ll 1$$

The major effect of $g_2(t)$ is actually found near $x = 0$, as we should also expect from the graph of $Ci(x)$ in Fig. 4.7.

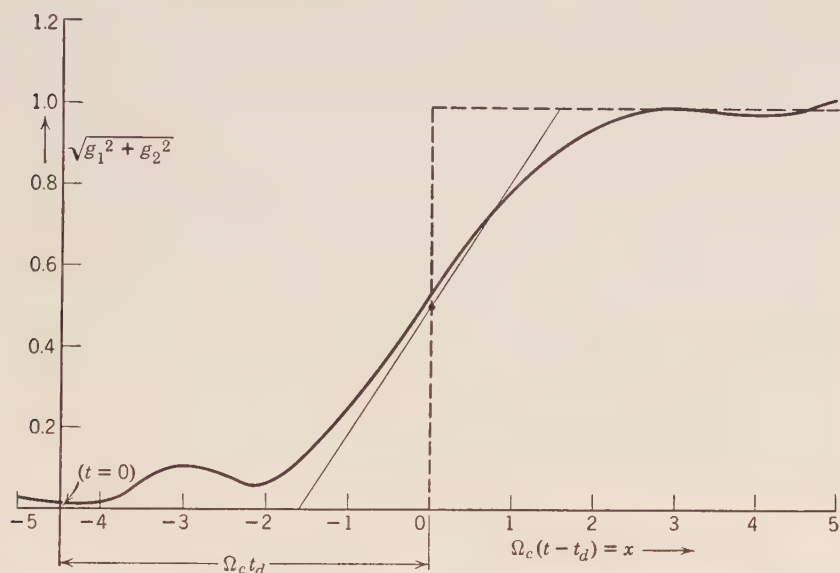


FIG. 4.11. Envelope of response of ideal low-pass filter to a suddenly applied carrier of constant amplitude and of frequency $\omega_0 = \frac{1}{2}\Omega_c$.

The phase angle η is by its definition variable with time, so that it constitutes an apparent frequency variation. We can compute this from the derivative of the argument

$$\frac{d}{dt} [\omega_0(t - t_d) + \psi + \eta] = \omega_0 + \frac{d\eta}{dt}$$

again in terms of the two functions $g_1(t)$ and $g_2(t)$.

We might also consider a band-pass filter with the characteristics given in Fig. 4.12. The analytic expressions for the filter character-

istics are identical with those given for the low-pass filter except that the frequency range is now defined by $\omega_1 < \omega < \omega_2$. Accordingly, if we apply a single frequency carrier wave defined as in (32) and with

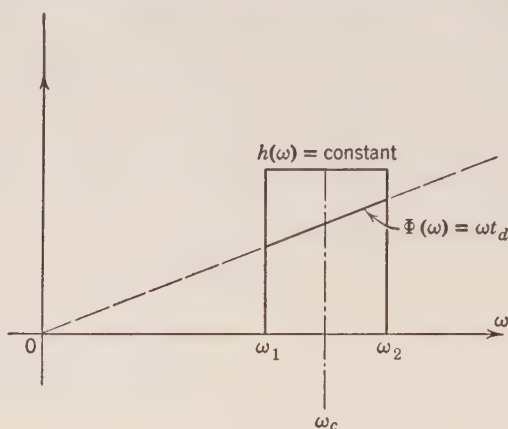


Fig. 4.12. Ideal band-pass filter characteristic. Pass band $\Delta\omega = \omega_2 - \omega_1$, center frequency $\omega_c = \frac{1}{2}(\omega_1 + \omega_2)$.

$\omega_1 < \omega_0 < \omega_2$, we obtain in complex notation for the response

$$\bar{v}_r(t) = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} A e^{-j\omega t_d} \frac{\bar{V}}{j(\omega - \omega_0)} e^{j\omega t} d\omega \quad (40)$$

where the pole $\omega = \omega_0$ is located on the path of integration. We can again introduce the variable $y = \omega - \omega_0$ as in (33) and obtain now the limits

$$\begin{aligned} \omega = \omega_1, & \quad y = \omega_1 - \omega_0 = -(\omega_0 - \omega_1) < 0 \\ \omega = \omega_2, & \quad y = \omega_2 - \omega_0 > 0 \end{aligned}$$

With $\tau = t - t_d$ we obtain now

$$\bar{v}_r(t) = \frac{A \bar{V}}{2\pi j} e^{j\omega_0 \tau} \int_{-j(\omega_0 - \omega_1)}^{+j(\omega_2 - \omega_0)} \frac{1}{jy} e^{jy\tau} d(jy) \quad (41)$$

which is quite similar to (34), and we could write the final response in the same form as (36) with the modifications dictated by the different limits.

Considerable simplification can be obtained if the carrier frequency ω_0 coincides with the center frequency ω_c of the transmission band in Fig. 4.12, i.e., if we have

$$\omega_0 = \omega_c = \frac{1}{2}(\omega_1 + \omega_2)$$

In this case we can also write, using $\Delta\omega = \omega_2 - \omega_1$ as the symbol for bandwidth,

$$\omega_2 - \omega_0 = +\frac{\Delta\omega}{2}, \quad -(\omega_0 - \omega_1) = -\frac{\Delta\omega}{2}$$

We have here then symmetrical limits for the integral, i.e., we obtain identically the integral form (15), so that we can write down the result at once from (24) as

$$\bar{v}_r(t) = A \bar{V} e^{j\omega_0(t-t_d)} \left[\frac{1}{2} + \frac{1}{\pi} Si \left(\frac{\Delta\omega}{2} (t - t_d) \right) \right] \quad (42)$$

The envelope of the a-c response for this so-called *symmetrical operation* of the band-pass filter is thus *identical* with the step response of the low-pass filter, if we replace the cutoff frequency Ω_c of the low-pass filter by the half bandwidth $(\Delta\omega)/2$ of the band-pass filter. It is rather natural that this should be so because we could assume that $\omega_0 = 0$, the zero carrier frequency, lies in the center of the low-pass filter band, extending this to $-\Omega_c < \omega < +\Omega_c$ in accordance with the Fourier transform concept. We can therefore use Fig. 4.8 with all positive values directly as the envelope of the symmetrical band-pass filter response.

For unsymmetrical operation we might rewrite the limits in the following manner

$$\begin{aligned} \omega_2 - \omega_0 &= \omega_c + \frac{\Delta\omega}{2} - \omega_0 = \frac{\Delta\omega}{2} - (\omega_0 - \omega_c) \\ -(\omega_0 - \omega_1) &= -\omega_0 + \omega_c - \frac{\Delta\omega}{2} = -\left(\frac{\Delta\omega}{2} + (\omega_0 - \omega_c) \right) \end{aligned}$$

We observe that these limits correspond to the limits in (34) if we again replace Ω_c of the low-pass filter by $(\Delta\omega)/2$ of the band-pass filter and introduce the relative carrier frequency $(\omega_0 - \omega_c)$ referred to the center frequency of the band-pass filter instead of the absolute carrier frequency ω_0 . With these substitutions we can take the solution of (41) as identical in form with (36). Moreover, we find the identical envelope as given in Fig. 4.11 for the special case of the carrier frequency being in the center of the right-half pass band, or $\omega_0 = \omega_c + [(\Delta\omega)/4]$. The interchange between low-pass and band-pass response characteristics is summarized in Table 4.1 which can be extended as a general correspondence.

TABLE 4.1
CORRESPONDENCE OF RESPONSE CHARACTERISTICS

	Ideal Low-Pass Filter	Ideal Band-Pass Filter
Band	$-\Omega_c < \omega < +\Omega_c$	$\omega_1 < \omega < \omega_2$
Bandwidth	$2\Omega_c$	$\Delta\omega = \omega_2 - \omega_1$
Center frequency	$\omega_c = 0$	$\omega_c = \frac{1}{2}(\omega_2 + \omega_1)$
Relative carrier frequency	ω_0	$\omega_0 - \omega_c$
Variable x as in Fig. 4.11	$\Omega_c(t - t_d)$	$\frac{\Delta\omega}{2}(t - t_d)$
y' in (36)	$(\Omega_c - \omega_0)(t - t_d)$	$\left(\frac{\Delta\omega}{2} - (\omega_0 - \omega_c)\right)(t - t_d)$ $= (\omega_2 - \omega_0)(t - t_d)$
y'' in (36)	$(\Omega_c + \omega_0)(t - t_d)$ $= [\omega_0 - (-\Omega_c)](t - t_d)$	$\left(\frac{\Delta\omega}{2} + (\omega_0 - \omega_c)\right)(t - t_d)$ $= (\omega_0 - \omega_1)(t - t_d)$
Special case of Fig. 4.11	$\omega_0 = \frac{\Omega_c}{2}$	$\omega_0 = \omega_c + \frac{\omega\Delta}{4}$

It is still of interest to evaluate the response of the ideal band-pass filter to a carrier frequency located outside the pass band, either $\omega_0 < \omega_1$ or $\omega_0 > \omega_2$. The integrals (40) and (41) remain unchanged in form; however, since either $\omega_2 - \omega_0 > 0$ and $\omega_1 - \omega_0 > 0$ or $\omega_0 - \omega_1 > 0$ and $\omega_0 - \omega_2 > 0$, the path of integration lies either wholly above the origin along the imaginary axis of the p -plane in Fig. 4.6 or wholly below the origin. This means that the pole $py = 0$ in (41) lies outside the path, the path itself is regular, and we can perform the integration directly. Suppose we choose the first alternative $\omega_0 < \omega_1$. We can then write the integral in (41)

$$\int_{j(\omega_1 - \omega_0)}^{j(\omega_2 - \omega_0)} = \int_{j(\omega_1 - \omega_0)}^{j\infty} - \int_{j\infty}^{j(\omega_2 - \omega_0)}$$

These integrals have been evaluated in (20), so that we can write the complete response

$$\begin{aligned} \bar{v}_r(t) &= \frac{A\bar{V}}{2\pi j} e^{j\omega_0 t} \left(-Ciy' + j\frac{\pi}{2} - jSiy' + Ciy'' - j\frac{\pi}{2} + jSiy'' \right) \\ &= A\bar{V}e^{j\omega_0 t} \frac{1}{2\pi} [(Siy'' - Siy') + j(Ciy' - Ciy'')] \end{aligned} \quad (43)$$

where

$$y' = (\omega_1 - \omega_0)(t - t_d), \quad y'' = (\omega_2 - \omega_0)(t - t_d)$$

The envelope is thus readily computed with the help of tables or of Fig. 4.7.

If the bandwidth $\Delta\omega$ is small and the carrier ω_0 not close to the pass band, then integration (34) can be simplified by replacing in good approximation

$$y = \omega - \omega_0 \quad \text{by } y_c = \omega_c - \omega_0$$

which is constant. Thus, integral (34) with the limits just stated for (43) gives

$$\frac{1}{y_c} \int_{\omega_1 - \omega_0}^{\omega_2 - \omega_0} e^{jy\tau} dy = \frac{1}{jy_c\tau} (e^{j(\omega_2 - \omega_0)\tau} - e^{j(\omega_1 - \omega_0)\tau})$$

which we can also write by introducing

$$\omega_{2,1} = \omega_c \pm \frac{\Delta\omega}{2}$$

in the form

$$\frac{1}{y_c\tau} e^{j(\omega_c - \omega_0)\tau} 2 \sin \frac{\Delta\omega}{2} \tau$$

The complete response is now, observing that the exponential factor in front of integral (34) cancels

$$\bar{v}_\tau(t) = \frac{A\bar{V}}{2\pi j} e^{j\omega_c(t-t_d)} \frac{\sin [(\Delta\omega)/2] (t - t_d)}{[(\Delta\omega)/2] (t - t_d)} \frac{\Delta\omega}{\omega_c - \omega_0} \quad (44)$$

The narrow-band filter responds to a frequency outside the pass band by oscillating at center frequency ω_c , with an amplitude variation given by $(1/x) \sin x$, building up to a maximum and then decreasing again. The envelope frequency is $(\Delta\omega)/2$ which is the difference frequency between the center and edge of the band, a very low frequency. The response itself is proportional to the band width but decreases with the distance of the carrier frequency from the center frequency.

Many other applications could easily be given but they would follow the same method of treatment, so they are reserved for the problems at the end of the chapter.

4.4 **Deviations From Ideal Characteristics**

The ideal characteristics assume constant amplitude and linear phase shift as functions of frequency as shown in Fig. 4.3 for the low-pass and in Fig. 4.12 for the band-pass filter, respectively. Though valuable for the study of systems and their appropriate components,

the response calculations have shown that other characteristics might lead to better performance depending upon the specific requirements on the system and the type of information it is designed to handle. It is therefore desirable to examine deviations from the so-called ideal characteristics and to compute their effects upon the network output.

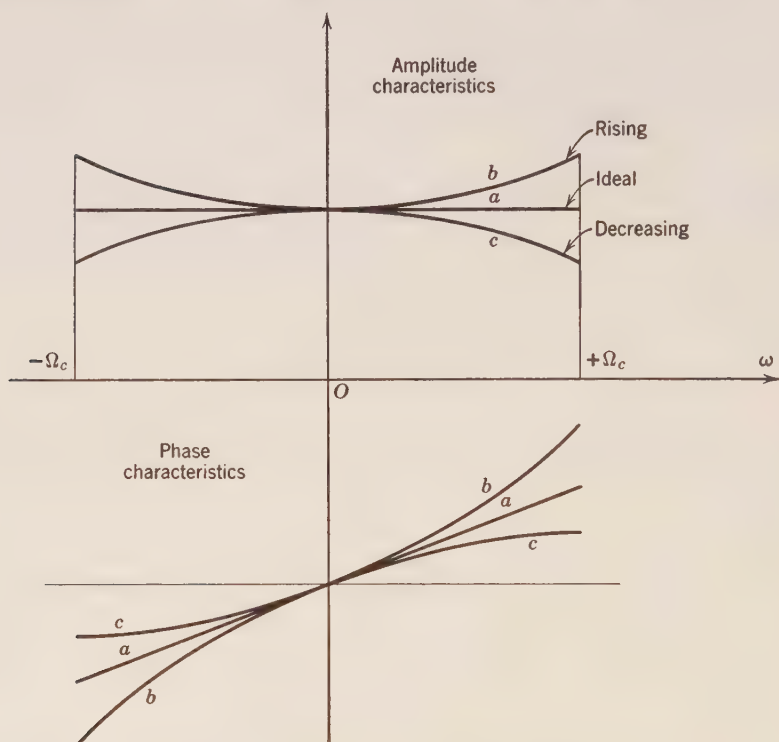


Fig. 4.13. Monotonic deviations from the ideal network characteristics a , a of the low-pass filter. Any combination of a , b , or c of amplitude with any a , b , or c of phase characteristic is feasible.

Several different approaches are possible, each entailing certain analytical difficulties.

We might first assume the ideal linear phase shift $\Phi(\omega) = \omega t_d$ combined with an amplitude characteristic given by

$$h(\omega) = A \left(1 \pm m \frac{\omega^2}{\Omega_c^2} \right) \quad (45)$$

where m is the measure of the maximum rise or decrease of amplitude at the edge of the band. Figure 4.13 illustrates this assumption with

curve a for phase and either b or c for amplitude for a low-pass filter. We introduce the parabolic shape because $h(\omega)$ is an even function in ω . The response to a step voltage input is given by (14) if we insert the frequency dependent factor to A from (45), or by

$$v_r(t) = \frac{1}{2\pi} \int_{-\Omega_c}^{+\Omega_c} A \left(1 \pm m \frac{\omega^2}{\Omega_c^2} \right) e^{-j\omega t_d} \frac{V_0}{j\omega} e^{j\omega t} d\omega \quad (46)$$

This can be resolved into two integrals by separating the terms in the parenthesis, the first integral being identical with (14) and therefore having the result given in (25). The second integral has the form

$$\pm \frac{mV_0A}{j2\pi\Omega_c^2} \int_{-\Omega_c}^{+\Omega_c} \omega e^{j\omega\tau} d\omega \quad (46a)$$

which gives upon integration by parts, calling $\omega = x$, $e^{j\omega\tau} d\omega = dy$

$$\pm V_0A \frac{m}{j2\pi\Omega_c^2} \left(\frac{1}{j\tau} (\Omega_c e^{j\Omega_c\tau} + \Omega_c e^{-j\Omega_c\tau}) - \frac{1}{(j\tau)^2} (e^{j\Omega_c\tau} - e^{-j\Omega_c\tau}) \right)$$

Upon collecting terms this becomes

$$\pm V_0A \frac{m}{\pi} \frac{1}{\Omega_c\tau} \left(\frac{\sin \Omega_c\tau}{\Omega_c\tau} - \cos \Omega_c\tau \right) = \pm V_0A \frac{m}{\pi} g(x)$$

where

$$g(x) = \frac{1}{x} \left(\frac{\sin x}{x} - \cos x \right), \quad x = \Omega_c(t - t_d) \quad (47)$$

is plotted in Fig. 4.14. This function is odd in x . The complete response (46) thus becomes

$$\frac{v_r(t)}{AV_0} = \mathfrak{R}(x) \pm \frac{m}{\pi} g(x) \quad (48)$$

where $\mathfrak{R}(x)$ is defined in (25) and plotted in Fig. 4.8. The second part depends on the measure m . For the value $m = 0.2$ we have plotted the total result (48) for both signs in Fig. 4.15. Clearly, a rising amplitude characteristic results in shorter build-up time—indicated by the straight lines—but also brings an increase in overshoot as comparison with Fig. 4.8 demonstrates. On the other hand, a decreasing amplitude characteristic results in appreciably longer build-up time but causes less overshoot.

If we apply the impulse $V_0TS_0(t)$ to these same network characteristics, we have to solve integral (30) whereby the amplitude A is

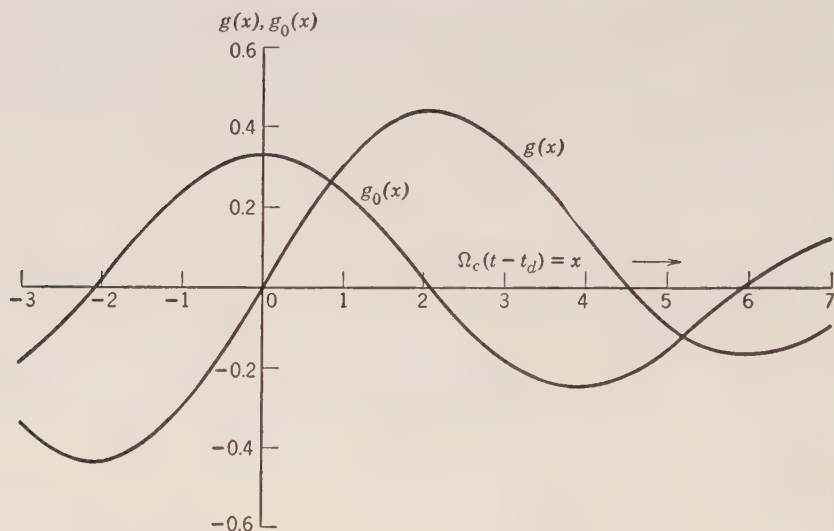


Fig. 4.14. The functions $g(x)$ from (47) and $g_0(x)$ from (50).

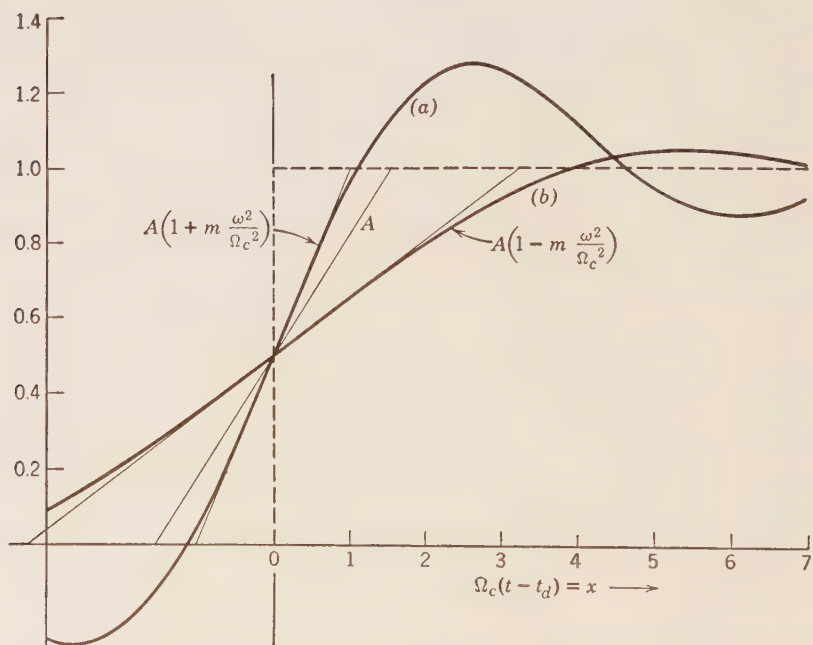


Fig. 4.15. Response to unit step of system with (a) rising amplitude characteristic, (b) falling amplitude characteristic. The phase shift is ideally linear.

replaced by (45), or

$$v_r(t) = \frac{1}{2\pi} \int_{-\Omega_c}^{+\Omega_c} A \left(1 \pm m \frac{\omega^2}{\Omega_c^2} \right) e^{-j\omega t_d} V_0 T e^{j\omega t} d\omega$$

Again we can split this integral by separating the terms in the parenthesis. The first part is identical with (29), the second part can be evaluated by integration by parts similar to (46a) except that we need two such processes in tandem to reduce the factor ω^2 to a constant. The final result becomes

$$\frac{v_r(t)}{A} = V_0 T \frac{\Omega_c}{\pi} \left(\frac{\sin x}{x} \pm m g_0(x) \right) \quad (49)$$

where

$$g_0(x) = \frac{\sin x}{x} - \frac{2}{x} g(x), \quad x = \Omega_c(t - t_d) \quad (50)$$

and $g(x)$ is the same function given in (47). The result, (49), is simply interpreted. We have the original response for the ideally flat characteristic as given in (29) and in appearance quite similar to the pulse response pictured in Fig. 4.10. Superimposed is the given function $g_0(x)$ from Fig. 4.14 which either adds or subtracts from Fig. 4.10 with their lines of symmetry coinciding. The rising amplitude characteristic (45) with the positive sign leads now to a sharpening of the impulse response, i.e., steeper rise, narrower base, and higher amplitude which are all desirable features for pulse response! The decreasing amplitude characteristic (45), with the negative sign, results in the opposite changes, i.e., lowers the amplitude and broadens the base of the response.

We might examine the impulse response for the combination of the ideally flat amplitude characteristic a in Fig. 4.13 with either b or c of the phase characteristics. Immediately, we are confronted with severe analytical difficulties. The phase characteristic should be an odd function so we might multiply the ideal $\Phi(\omega) = \omega t_d$ by the same factor as the amplitude in (45). However, this would lead to ω^3 in the exponent of the phase characteristic and we can see no hope for solving the resultant integrals in closed form. We might try as a somewhat simpler expression

$$\Phi(\omega) = \omega t_d \pm m \left(\frac{\omega}{\Omega_c} \right)^2 \quad \omega \gtrless 0 \quad (51)$$

where we need to change the sign of the deviation term in order to assure odd symmetry in ω . Formulation (51) describes curve b of the

phase characteristic in Fig. 4.13; if we interchange the signs we obtain curve *c*. Introducing (51) into (30) which gives the impulse response for the ideal characteristics, we see that we need to write two integrals

$$v_r(t) = \frac{1}{2\pi} \int_0^{\Omega_c} A \exp \left[-j \left(\omega t_d + m \frac{\omega^2}{\Omega_c^2} \right) \right] V_0 T e^{j\omega t} d\omega \\ + \frac{1}{2\pi} \int_{-\Omega_c}^0 A \exp \left[-j \left(\omega t_d - m \frac{\omega^2}{\Omega_c^2} \right) \right] V_0 T e^{j\omega t} d\omega \quad (52)$$

Replacing ω by $-\omega$ in the second integral makes all exponents conjugate imaginary to those in the first integral and permits the same limits so that we can contract the integrand into

$$\frac{1}{\pi} A V_0 T \int_0^{\Omega_c} \cos \left(\omega(t - t_d) - m \frac{\omega^2}{\Omega_c^2} \right) d\omega \quad (52a)$$

It is entirely impossible here to separate the response of the ideal, linear phase characteristic from that of the deviation term. This is rather important because it illustrates the much greater involvement of the phase effect, a fact that has only been stressed recently.* To bring integral (52a) into a standard form, we need to complete the square in the argument of the cosine, where we can disregard the negative sign since the cosine is insensitive to the change of sign

$$\left[\sqrt{m} \left(\frac{\omega}{\Omega_c} \right) - \frac{\Omega_c(t - t_d)}{2\sqrt{m}} \right]^2 - \left(\frac{\Omega_c(t - t_d)}{2\sqrt{m}} \right)^2 = u^2 - u_0^2 \quad (53)$$

where

$$u_0 = \frac{x}{2\sqrt{m}} = \frac{\Omega_c(t - t_d)}{2\sqrt{m}} \quad (53a)$$

The quantity x is directly proportional to time and we have used it consistently in the graphs of the functions of interest. With the new variable u , integral (52a) takes the form

$$v_r(t) = \frac{1}{\pi} A V_0 T \frac{\Omega_c}{\sqrt{m}} \int_{u=-u_0}^{\sqrt{m}-u_0} \cos(u^2 - u_0^2) du \quad (54)$$

Actually, the standard functions which can be used here are the Fresnel integrals defined by†

* M. J. DiToro, "Phase and Amplitude Distortion in Linear Networks," *Proc. I.R.E.*, **36**, 24-36 (1948); also doctoral dissertation, Polytechnic Institute of Brooklyn, June 1946.

† Jahnke-Emde, *op. cit.*, pp. 34-37.

$$C(z) - jS(z) = \int_0^z \frac{e^{-jy}}{\sqrt{2\pi y}} dy = \frac{1}{\sqrt{2\pi}} \int_0^z \frac{\cos y}{\sqrt{y}} dy - j \frac{1}{\sqrt{2\pi}} \int_0^z \frac{\sin y}{\sqrt{y}} dy \quad (55)$$

If we introduce

$$y = \frac{\pi}{2} u^2, \quad dy = \pi u du$$

the complex integral becomes

$$C(u) - jS(u) = \int_0^u \frac{e^{-j\frac{\pi}{2}u^2}}{\pi u} \pi u du = \int_0^u e^{-j\frac{\pi}{2}u^2} du = \int_0^u j^{-u^2} du \quad (56)$$

Using this identity in (55) and separating real and imaginary parts, we have the equivalent standard definitions

$$C(u) = \int_0^u \cos\left(\frac{\pi}{2} u^2\right) du = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{\frac{2}{\pi}}u} \cos u^2 du \quad (57)$$

$$S(u) = \int_0^u \sin\left(\frac{\pi}{2} u^2\right) du = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{\frac{2}{\pi}}u} \sin u^2 du \quad (57)$$

The tables in Jahnke and Emde contain values of both standard definitions (55) and (57). We can expand the cosine in (54) and identify the integrals of the variable u from (57), keeping account of the factor $\pi/2$ in the integrand, so that

$$\begin{aligned} \frac{v_r(t)}{A} = V_0 T \frac{\Omega_c}{\sqrt{2\pi m}} & \left\{ \left[C\left(\sqrt{\frac{2}{\pi}}(\sqrt{m} - u_0)\right) + C\left(\sqrt{\frac{2}{\pi}}u_0\right) \right] \cos u_0^2 \right. \\ & \left. + \left[S\left(\sqrt{\frac{2}{\pi}}(\sqrt{m} - u_0)\right) + S\left(\sqrt{\frac{2}{\pi}}u_0\right) \right] \sin u_0^2 \right\} \quad (58) \end{aligned}$$

A similar result is given in Küpfmüller,^{A10} p. 82, where, however, the upper limit in (52a) is chosen as ∞ , so that we need to substitute $u \rightarrow \infty$ and therefore

$$C\left(\sqrt{\frac{2}{\pi}}(\sqrt{m} - u_0)\right) \rightarrow C(\infty) = \frac{1}{2}$$

$$S\left(\sqrt{\frac{2}{\pi}}(\sqrt{m} - u_0)\right) \rightarrow S(\infty) = \frac{1}{2}$$

The wave shape of the response is indicated in Fig. 4.16. As evident from (58), the principal periodicity is contributed by $\cos u_0^2$ and $\sin u_0^2$, both of which exhibit quadratically increasing frequency. We recognize some similarity with Fig. 4.10 but the build-up is accelerated and the deviation of the phase characteristic from linearity causes high frequency oscillations of relatively low amplitude except in direct proximity to the main pulse.

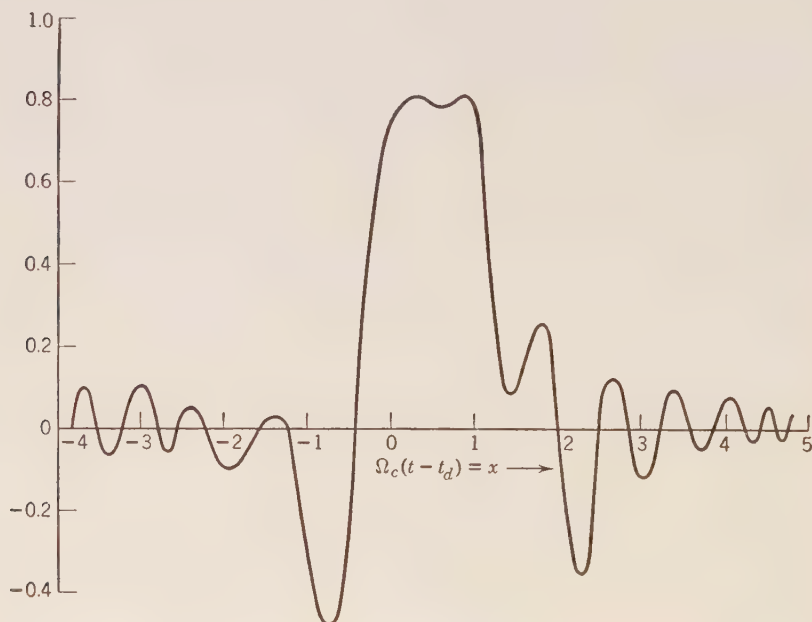


Fig. 4.16. Impulse response of system with flat amplitude characteristic a and phase characteristic b of Fig. 4.13.

We can now readily see that more general cases of combined amplitude and phase deviations from the ideal characteristics become very involved. A number of special tractable cases are carried through in Küpfmüller,^{A10} chapter IV, and in nonmathematical form in Cherry,^{A4} section 5.

We should, of course, refer here again to the method* of "paired echos," briefly discussed in section 2.7. Suppose we consider a low-pass filter with the general network characteristic

$$A(\omega)e^{-j\Phi(\omega)} = A(\omega) \cos \Phi(\omega) - jA(\omega) \sin \Phi(\omega) \quad (59)$$

* H. Wheeler, "The Interpretation of Amplitude and Phase Distortion in Terms of Paired Echos," *Proc. I.R.E.*, **27**, 359-384 (1939). Also Cherry,^{A4} pp. 296-302.

We can develop this characteristic into a complex Fourier series as discussed in Vol. I, section 6.2, taking the total frequency band $2\Omega_c$ as the fundamental period. This will give for a finite number of terms

$$A(\omega)e^{-j\Phi(\omega)} = \sum_{n=-N}^{+N} C_n' e^{j\frac{n\pi\omega}{\Omega_c}} \quad (60)$$

where the coefficients are found by

$$C_n' = \frac{1}{2\Omega_c} \int_{-\Omega_c}^{+\Omega_c} A(\omega)e^{-j\Phi(\omega)} e^{-j\frac{n\pi\omega}{\Omega_c}} d\omega \quad (61)$$

We have stressed that $A(\omega)$ must be an even function of ω , and $\Phi(\omega)$ an odd function, so that the coefficients C_n' will definitely be real. To establish closer relationship with the previously treated ideal characteristics, we might change this expansion slightly by segregating the linear phase

$$(A_0 e^{-j\omega t_d}) \frac{A(\omega)}{A_0} e^{-j\Delta\Phi(\omega)}$$

where $\Delta\Phi(\omega)$ is the deviation from the linear phase characteristic and must again be an odd function of ω . Expanding the factor outside the parenthesis as in (60), we obtain

$$A(\omega)e^{-j\Phi(\omega)} = A_0 e^{-j\omega t_d} \sum_{n=-N}^{+N} C_n e^{j\frac{n\pi\omega}{\Omega_c}} \quad (62)$$

where the C_n are determined in analogous manner to (61).

The response to any applied source function $v_s(t)$ with Fourier transform $\mathcal{F}v_s(t)$ is now given by

$$v_r(t) = \frac{A_0}{2\pi} \int_{-\Omega_c}^{+\Omega_c} e^{-j\omega t_d} \left(\sum_{n=-N}^{+N} C_n e^{j\frac{n\pi\omega}{\Omega_c}} \right) \mathcal{F}v_s(t) e^{j\omega t} d\omega \quad (63)$$

For the actual evaluation we need to specify $v_s(t)$ because of the finite frequency limits of the integral. If we had complete Fourier transform integration, we could write the response directly as in (2.123). Suppose we take a step source function $V_0 I$, with Fourier transform $V_0/j\omega$; then (63) can be rewritten

$$v_r(t) = \frac{A_0 V_0}{2\pi j} \sum_{n=-N}^{+N} C_n \int_{-j\Omega_c}^{+j\Omega_c} \frac{1}{j\omega} e^{j\omega \tau_{nd}} d(j\omega) \quad (64)$$

where the integral is exactly the same as (15), except that we have here

$$\tau_n = t - t_d + \frac{n\pi}{\Omega_c} \quad (64a)$$

We can therefore take the answer from (25)

$$\frac{v_r(t)}{A_0 V_0} = \sum_{n=-N}^{+N} C_n \left(\frac{1}{2} + \frac{1}{\pi} Si(\Omega_c \tau_n) \right) \quad (65)$$

The result is in very simple form, easy to interpret. For $n = 0$ we obtain a term identical with (25) in every respect; we have essentially the response of the ideal low-pass filter as shown in Fig. 4.8. For each positive and negative n we obtain, respectively, an advanced and a delayed "echo" with time spacing $\pm[(\pi n)/\Omega_c]$ with respect to the original $n = 0$, and with amplitude C_{+n} and C_{-n} defined by the Fourier analysis of the network characteristic itself. The designation as a method of "paired echos" is now quite obvious. If series (62) is quickly convergent, this method is quite instructive and useful, particularly if it is referred to the ideal characteristics because the deviations are then the obvious causes of the echos.

We could, of course, use this method to study the effect of some periodic ripple superimposed upon the amplitude or phase characteristic.

4.5 Relations Between Network Characteristics

We have accepted in the foregoing discussions the nonphysical behavior of the abstract network with idealized characteristics because we could gain rather broad relations for the transient response of systems which are valuable in themselves. We suspected at the time that this nonphysical behavior must be caused by an inadmissible simultaneous choice of amplitude and phase characteristics. Actually, the close interrelation of network functions and the theory of functions of a complex variable, brought to light by the increasing use of the Laplace transform method, should call for a critical examination of the analytical properties of physically realizable networks. The first extensive study was published by M. Bayard* and many additional relations were deduced by Bode,^{B2} chapters 13 and 14.

We might start from the fact that any regular driving point impedance $Z(p)$ of a passive linear network cannot have poles in the right

* M. Bayard, "Relations Between the Real and Imaginary Parts of Impedances and Determination of Impedances when one of these Parts is given" [French], *Revue gén. élec.*, **37**, 659-664 (1935).

half of the p -plane. If there are poles along the imaginary axis of $j\omega$, including the possibility of poles at $p = 0$ and at $p = j\infty$, we will remove these from $Z(p)$ so that we can concentrate on

$$Z_0(p) = Z(p) - \left(a_1 p + \frac{b_1}{p} + \sum_{\alpha} \frac{c_{\alpha} p}{p^2 + \omega_{\alpha}^2} \right) \quad (66)$$

a function completely regular in the entire right-half p -plane. Two-terminal networks described by impedance functions of the type $Z_0(p)$ have been called *minimum reactance networks* by Bode,^{B2} chapters 7.8 and 9.3. Obviously, we can consider an admittance function $Y(p)$ in the same manner and arrive then at a *minimum susceptance network*. Because the function $Z_0(p)$ is regular in the entire right-half p -plane and on its boundaries, we can write it in terms of the complex variable $p = \delta + j\omega$

$$Z(\delta + j\omega) = R(\delta, \omega) + jX(\delta, \omega) \quad (67)$$

with the Cauchy-Riemann relations satisfied everywhere except at singularities

$$\frac{\partial R}{\partial \delta} = \frac{\partial X}{\partial \omega}, \quad \frac{\partial R}{\partial \omega} = -\frac{\partial X}{\partial \delta} \quad (67a)$$

For $\delta \rightarrow 0$, we have the conventional a-c impedance function

$$Z(j\omega) = R(\omega) + j(X\omega) \quad (68)$$

where $R(\omega)$ is the resistance and $X(\omega)$ the reactance function in the conventional sense.

If we now have given $R(\omega)$, i.e., the real part of the impedance function is prescribed, we can evaluate the complex impedance function and thus the reactance part by simple application of relations known from the theory of functions of a complex variable. First of all, we can construct to $R(\omega)$ the generalized function $R(\delta, \omega)$ as the real part of an analytic function by invoking two-dimensional potential theory. Because of the Cauchy-Riemann equations (67a), $R(\delta, \omega)$ and $X(\delta, \omega)$ satisfy the Laplace differential equations

$$\begin{aligned} \frac{\partial^2 R}{\partial \delta^2} + \frac{\partial^2 R}{\partial \omega^2} &= 0 \\ \frac{\partial^2 X}{\partial \delta^2} + \frac{\partial^2 X}{\partial \omega^2} &= 0 \end{aligned} \quad (69)$$

as second differentiations of (67a) demonstrate. This clearly means that both $R(\delta, \omega)$ and $X(\delta, \omega)$ are so-called *harmonic functions*, and

since they constitute real and imaginary parts of the same analytic function $Z(p)$ they are called *conjugate functions*.^{*} To solve for the potential within a regular region if it is known everywhere on the boundary is the classical problem of Dirichlet. In two dimensions this solution is an application of Cauchy's integral theorem, given in Appendix (5.18) and illustrated in Fig. A-5.5.

The simplest classical solution for the real potential problem has been developed by Poisson[†] for the circle. If the potential Φ is known along the circle of radius a in the z -plane of Fig. 4.17 as a function of the angle ϕ' counted along this circle, $\Phi(\phi')$, then at any point Z with

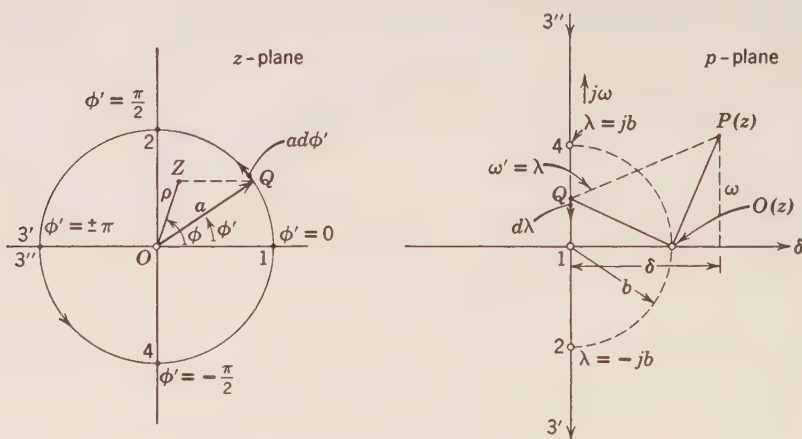


Fig. 4.17. Relation of regular impedance function to potential theory. Conformal mapping of interior of circle $ae^{j\phi'}$ in z -plane upon right half of p -plane.

polar coordinates $\rho e^{j\phi}$ the potential value $\Phi(\rho, \phi)$ is found by Poisson's integral

$$\Phi(\rho, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2 - \rho^2}{a^2 - 2a\rho \cos(\phi - \phi') + \rho^2} \Phi(\phi') d\phi' \quad (70)$$

The complex analytical potential

$$P(z) = \Phi(\rho, \phi) + j\Xi(\rho, \phi)$$

of which Φ is the real part, can also be found directly from $\Phi(\phi')$ for the circle by the integral developed by Schwarz[†]

$$P(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{ae^{j\phi'} + z}{ae^{j\phi'} - z} \Phi(\phi') d\phi' \quad (71)$$

^{*} E. Weber, *Electromagnetic Fields: Theory and Application*, Vol. I, *Mapping of Fields*, chapter 7, particularly pp. 279–282, Wiley, New York, 1950.

[†] *Ibid.*, pp. 362–365.

This we can translate at once into our impedance function with complex variable z , if the real part is known along the circle $ae^{j\phi'}$, namely

$$Z(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{ae^{j\phi'} + z}{ae^{j\phi'} - z} R(\phi') d\phi' = R(\rho, \phi) + jX(\rho, \phi) \quad (72)$$

Actually, we would like to know $Z(p)$ in the complex p -plane with $R(\delta, \omega)$ given along the imaginary axis as $R(\omega)$. This can readily be obtained if we map the circle $ae^{j\phi'}$ from the z -plane conformally into the imaginary axis $j\omega$ of the p -plane such that the points 1, 2, 3', 3'', 4 correspond to each other as shown in Fig. 4.17. Progressing along the circle in the z -plane in the mathematically positive sense, the area mapped is on the left and this corresponds in the p -plane to the right half $\delta \geq 0$. The mapping function is

$$p = b \frac{a - z}{a + z}, \quad z = -\frac{p - b}{p + b} a \quad (73)$$

If we insert the coordinates of the circle, $z = ae^{j\phi'}$, we find

$$p = -jb \frac{\sin \phi'}{1 + \cos \phi'} = j\omega \quad (74)$$

and the correspondence

$$\begin{aligned} \phi' = 0, & \quad e^{j\phi'} = 1, & \quad p = 0 \\ \phi' = \pm \frac{\pi}{2}, & \quad e^{j\phi'} = \pm j, & \quad p = \mp jb \\ \phi' = \pm \pi, & \quad e^{j\phi'} = -1, & \quad p = \mp j\infty \end{aligned}$$

We can now introduce z from (73) into the integral (72) observing that for $\rho = a$ we have

$$ae^{j\phi'} = -\frac{j\omega' - b}{j\omega' + b} a$$

and therefore

$$je^{j\phi'} d\phi' = \frac{-2jb}{(j\omega' + b)^2} d\omega'$$

or with $e^{j\phi'}$ from the preceding

$$d\phi' = -\frac{2b}{(b^2 + \omega'^2)} d\omega'$$

We have used ω' to designate again the values along the boundary, i.e., the imaginary axis, because it corresponds to the circle $ae^{j\phi'}$ along

which $R(\phi')$ is given. To avoid confusion, we shall use from now on $\omega' = \lambda$ as indicated in Fig. 4.17. Thus, integral (72) transforms, after taking the limits from $-\pi$ to $+\pi$ rather than from 0 to 2π , into

$$Z(p) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{b^2 - j\lambda p}{b(p - j\lambda)} R(\lambda) \frac{2b}{(b^2 + \lambda^2)} d\lambda \quad (75)$$

The new limits should actually be $+\infty$ to $-\infty$ but the negative sign from $d\phi'$ permits interchange into the conventional direction of integration. We can still rationalize the first denominator and actually separate (75) into two integrals

$$Z(p) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{p}{p^2 + \lambda^2} R(\lambda) d\lambda + \frac{j}{\pi} \int_{-\infty}^{+\infty} \frac{\lambda(b^2 - p^2)}{(b^2 + \lambda^2)(p^2 + \lambda^2)} R(\lambda) d\lambda \quad (76)$$

At this point it is important to recall that for any realizable impedance function (68), $R(\omega)$ must be an even function and $X(\omega)$ an odd function of ω . This physical restriction for network functions very conveniently drops the second integral in (76) because it has an odd integrand in λ as is quite obvious by the extra factor λ . We thus arrive at the very simple relation

$$Z(p) = \frac{2}{\pi} \int_0^{+\infty} \frac{p}{p^2 + \lambda^2} R(\lambda) d\lambda \quad (77)$$

where we utilized the fact that the integrand was an even function and adjusted the lower limit. This gives now the *complete complex frequency impedance function* in terms of the known real part along the real frequency axis (imaginary axis $j\omega$ in the p -plane).

Upon writing $p = \delta + j\omega$ and separating real and imaginary parts, we obtain from (77) in accordance with (67) the functions $R(\delta, \omega)$ and $X(\delta, \omega)$. If in the latter we let $\delta \rightarrow 0$, we obtain at once—or also directly from (77)

$$X(\omega) = \frac{2}{\pi} \int_0^{\infty} \frac{\omega}{\lambda^2 - \omega^2} R(\lambda) d\lambda \quad (78)$$

which again is a simple expression. The only disadvantage of this form is the fact that for $\lambda = \omega$ the integrand becomes infinite. To adjust this, we observe that

$$\int_0^{+\infty} \frac{d\lambda}{\lambda^2 - \omega^2} = \frac{1}{2\omega} \ln \left. \frac{\lambda - \omega}{\lambda + \omega} \right|_0^{+\infty} = 0 \quad (79)$$

This integrand still becomes infinite as $\lambda = \pm\omega$, but we can evaluate, disregarding the constant factor $1/2\omega$

$$\lim_{\epsilon \rightarrow 0} \left(\int_0^{\omega-\epsilon} + \int_{\omega+\epsilon}^{\infty} \right) = \lim \left(\ln \frac{-\epsilon}{2\omega - \epsilon} - \ln(-1) + 0 - \ln \frac{\epsilon}{2\omega + \epsilon} \right)$$

The first and last terms in the brackets combine to

$$\ln(-1) + \ln \frac{2\omega + \epsilon}{2\omega - \epsilon}$$

so that $\ln(-1)$ cancels and the limit $\epsilon \rightarrow 0$ leads to (79).

We utilize this by introducing this integrand into (78) with the factor $\omega R(\omega)$ which does not participate in the integration so that we get

$$X(\omega) = \frac{2}{\pi} \int_0^{\infty} \frac{\omega}{\lambda^2 - \omega^2} [R(\lambda) - R(\omega)] d\lambda \quad (80)$$

which now remains finite at $\lambda = \pm\omega$. This is one of the *fundamental relations connecting real and imaginary parts* of the realizable network function. We have shown this in detail for the impedance function, but obviously the same holds for admittance functions with appropriate substitutions.

The reverse relationship, namely finding $R(\omega)$ for a given $X(\omega)$, can proceed quite similarly. We can start from the fact that $X(\delta, \omega)$ is a potential function in view of (69), thus can be evaluated like (70) within the circle $ae^{j\phi'}$ from its values along the circular periphery. We can at once construct a complex potential function (71) and thus a complex impedance function like (72), but now its real value must be the reactance function. This we can achieve if we multiply (67) by $-j$

$$-jZ(p) = X(\delta, \omega) - jR(\delta, \omega)$$

so that we must identify here $P = -jZ$. We now have, if we go directly to (76)

$$\begin{aligned} -jZ(p) = & \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{p}{p^2 + \lambda^2} X(\lambda) d\lambda \\ & + \frac{j}{\pi} \int_{-\infty}^{+\infty} \frac{\lambda(b^2 - p^2)}{(b^2 + \lambda^2)(p^2 + \lambda^2)} X(\lambda) d\lambda \end{aligned} \quad (81)$$

At this point we need to recall that the reactance function of any realizable network must be an odd function of ω , or λ as we call the variable. This physical restriction now drops the first integral in (81) because of the symmetrical limits. Being left with the second integral,

we can cancel j and write the identity in the numerator

$$b^2 - p^2 = (b^2 + \lambda^2) - (p^2 + \lambda^2)$$

which permits separation into the simpler integrals

$$Z(p) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\lambda}{p^2 + \lambda^2} X(\lambda) d\lambda + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\lambda}{b^2 + \lambda^2} X(\lambda) d\lambda \quad (82)$$

This is the exact equivalent of (77) and gives the *complete complex frequency impedance function* in terms of the known imaginary part along the real frequency axis (imaginary axis $j\omega$ in the p -plane). We observe that the second part is actually independent of p and therefore of ω . We must identify this with the d-c resistance value

$$Z(0) = R(0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\lambda}{b^2 + \lambda^2} X(\lambda) d\lambda \quad (83)$$

given here explicitly in terms of the prescribed reactance function. This leads finally, if we take into account the symmetry of the integrand, to

$$Z(p) = R(0) - \frac{2}{\pi} \int_0^{+\infty} \frac{\lambda}{p^2 + \lambda^2} X(\lambda) d\lambda \quad (84)$$

Introducing $p = \delta + j\omega$ and separating real and imaginary parts, we obtain from (84) $R(\delta, \omega)$ and $X(\delta, \omega)$. If in the former we let $\delta \rightarrow 0$ we obtain at once, also directly from (84)

$$R(\omega) = R(0) - \frac{2}{\pi} \int_0^{\infty} \frac{\lambda}{\lambda^2 - \omega^2} X(\lambda) d\lambda \quad (85)$$

If we like, we may again use (79) to amplify the integrand so that we avoid the infinity at $\lambda = \omega$

$$R(\omega) = R(0) - \frac{2}{\pi} \int_0^{\infty} \frac{\lambda X(\lambda) - \omega X(\omega)}{\lambda^2 - \omega^2} d\lambda \quad (86)$$

which is a form derived differently by Bayard, *loc. cit.*, and Bode.^{B2} The close relationship between realizable network functions and electrostatic potential theory emphasized in this derivation has been utilized in the electrolytic tank in the design of networks with desirable frequency characteristics.*

* E. C. Cherry, "Application of the Electrolytic Tank Techniques to Network Synthesis," *Proc. Symp. on Modern Network Synthesis*, Polytechnic Institute of Brooklyn, New York, 1952. Also H. A. Wheeler, "The Potential Analogue Applied to the Synthesis of Stagger-Tuned Filters," *I.R.E. Trans., Circuit Theory*, CT-2, No. 1, 86-96 (1955).

If $Z(p)$ represents a transfer impedance, then our requirements of regularity in the entire right-half p -plane specify a so-called *minimum phase network*, Bode,^{B2} section 7.8, but the relations between real and imaginary parts would be the same as given in (80) and (86). Of course, we could substitute the appropriate admittance function. Frequently we use, however, amplitude and phase functions and for this purpose we study not $Z(p)$ itself but its logarithm, e.g., in the notation of Bode,^{B2} p. 230

$$\begin{aligned}\theta(j\omega) &= \ln \frac{Z_T}{2 \sqrt{R_1 R_2}} = A(\omega) + jB(\omega) \\ &= \ln \frac{|Z_T|}{2 \sqrt{R_1 R_2}} + j\psi(\omega)\end{aligned}\quad (87)$$

where Z_T is the transfer impedance and R_1 and R_2 the terminating resistances, $A(\omega)$ the attenuation function, and $B(\omega) = \psi(\omega)$ the phase function if we write Z_T in polar form. We have chosen in (3) as transfer function

$$H(j\omega) = |H(j\omega)|e^{-j\Phi(\omega)}$$

so that with $h(\omega) = |H(j\omega)|$

$$\ln H(j\omega) = \ln h(\omega) - j\Phi(\omega) \quad (88)$$

which will make the phase delay a positive angle; this convention is most frequently found in dealing with the transient response of networks, e.g., Cherry,^{A4} p. 84, Guillemin,^{A8} p. 475, Küpfmüller,^{A10} p. 33. Because

$$\ln \frac{f(p)}{g(p)} = \ln f(p) - \ln g(p)$$

we must require in this case that there be no zeros in the right-half plane since they become singularities and thus violate the original assumption of regularity of the complex function.

With the notation in (88) we may adapt at once relation (80) expressing the imaginary part or phase function in terms of the real part or amplitude function

$$\Phi(\omega) = -\frac{2\omega}{\pi} \int_0^\infty \frac{\ln h(\lambda)}{\lambda^2 - \omega^2} d\lambda \quad (89)$$

and conversely, (85)

$$\ln h(\omega) = \ln h(0) + \frac{2}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2 - \omega^2} \Phi(\lambda) d\lambda \quad (90)$$

If we prefer the amplified expressions to remove, at least formally, the indeterminateness of the integrand at $\lambda = \omega$, we have for (80)

$$\Phi(\omega) = -\frac{2\omega}{\pi} \int_0^\infty \frac{\ln h(\lambda) - \ln h(\omega)}{\lambda^2 - \omega^2} d\lambda \quad (91)$$

and for (86)

$$\ln h(\omega) - \ln h(0) = \frac{2}{\pi} \int_0^\infty \frac{\lambda \Phi(\lambda) - \omega \Phi(\omega)}{\lambda^2 - \omega^2} d\lambda \quad (92)$$

4.6 Approximations to Distortionless Transmission

We have established that there exist unique relations between the real and imaginary parts of complex network functions regular in the entire right-half p -plane. These network functions can be impedances, admittances, transfer functions, or their logarithms. When we introduced idealized network functions we decidedly assumed regular forms but we selected both amplitude and phase functions, thus violating the relations deduced. We might now inquire first as to what we should have assumed, and second, in what direction we would need to modify the characteristics to obtain nearly ideal responses. Unfortunately, there can be no positive answer to the second question, because signal characteristics and network performance are interdependent.

The idealized low-pass filter had the characteristics

$$h(\omega) = A, \quad \Phi(\omega) = \omega t_d, \quad -\Omega_c < \omega < +\Omega_c$$

as shown in Fig. 4.3. Let us retain the flat amplitude characteristic and find the compatible phase characteristic which would lead to a realizable impedance function. We might use (89) directly which gives here

$$\Phi(\omega) = -\frac{2\omega}{\pi} (\ln A) \int_0^{\Omega_c} \frac{d\lambda}{\lambda^2 - \omega^2} \quad (93)$$

The integral must be evaluated with due regard to what we said in connection with (79), i.e., we evaluate

$$\lim_{\epsilon \rightarrow 0} \int_0^{\omega - \epsilon} + \int_{\omega + \epsilon}^{\Omega_c} = \frac{1}{2\omega} \ln \frac{\Omega_c - \omega}{\Omega_c + \omega}$$

because the contributions at all limits except Ω_c cancel out. We thus have for $\omega < \Omega_c$, inverting the logarithm and

$$\Phi(\omega) = \frac{1}{\pi} (\ln A) \ln \frac{\Omega_c + \omega}{\Omega_c - \omega} \quad \omega < \Omega_c \quad (94)$$

For small values of ω we can approximate

$$\ln \frac{1 + (\omega/\Omega_c)}{1 - (\omega/\Omega_c)} \approx 2 \ln \left(1 + \frac{\omega}{\Omega_c} \right) \approx 2 \frac{\omega}{\Omega_c}$$

and therefore

$$\Phi(\omega) \approx \frac{2}{\pi} (\ln A) \frac{\omega}{\Omega_c} \quad \omega \ll \Omega_c \quad (95)$$

We have linear phase shift at low frequencies as we had assumed in the idealized characteristic, and we can even determine the compatible delay time t_d as the slope from (95)

$$t_d = \frac{2}{\pi \Omega_c} \ln A \quad (95a)$$

The smaller Ω_c and the larger A , the longer will be the delay time. As we approach the edge of the band, however, $\Phi(\omega)$ increases to a logarithmic infinity which is characteristic for any discontinuity in the assumed attenuation characteristic. This demonstrates a wide discrepancy between Fig. 4.3 and the compatible phase values.

We had disregarded the phase characteristic beyond $\omega = \Omega_c$ in Fig. 4.3. Actually, (93) is presumably valid for any ω , so that for $\omega > \Omega_c$ we obtain

$$\Phi(\omega) = \frac{2\omega}{\pi} (\ln A) \int_0^{\Omega_c} \frac{d\lambda}{\omega^2 - \lambda^2} = \frac{1}{\pi} (\ln A) \ln \frac{\omega + \Omega_c}{\omega - \Omega_c} \quad \omega > \Omega_c \quad (96)$$

where the integrand is regular for all $0 < \lambda < \Omega_c$ so that the integral can easily be evaluated. Outside the pass band, then, the phase function continues, declining logarithmically from the infinite value at $\omega = \Omega_c$ to zero as $\omega \rightarrow \infty$. Obviously, however, if we let $\Omega_c \rightarrow \infty$, the ideal characteristics become mutually compatible because we can make the assumption of (95) for any finite value of ω . The time delay becomes very small as (95a) indicates. Thus, the wider the transmission band, the closer we approach distortionless transmission, which is natural. A good illustration is the degree of resolution between two closely spaced pulses as a function of the bandwidth of an idealized low-pass filter.*

Actually, the frequency spectra for the pulse group, Fig. 1.2, given in Figs. 1.5a and b and for the pulse group, Fig. 1.6, given in Fig. 1.9 indicate fairly clearly that the major contribution and the character-

* W. L. Sullivan, "Analysis of Systems with Known Transmission-Frequency Characteristics by Fourier Integrals," *Elec. Eng.*, **61**, 248-256 (1942).

istic differences in amplitude appear to be concentrated in the band $f < 1/T$, where T is the duration of the pulse, whereas the major contributions to the phase characteristics appear to extend further but depend strongly upon the signal shape. The importance of the phase function for the distortionless transmission has been particularly recognized* in connection with pulse-type signals where transient response becomes very important. Selecting a narrow transmission band for economic reasons, it then becomes important to shape the amplitude characteristic† and to select a pulse form which has narrowly limited frequency spectra such as the Gauss error function type.‡

If, on the other hand, we start from a linear phase characteristic over the band as in Fig. 4.3, we find for the amplitude characteristic with (90) and $\Phi(\lambda) = \lambda t_d$

$$\ln \frac{h(\omega)}{h(0)} = \frac{2}{\pi} t_d \int_0^{\Omega_c} \frac{\lambda^2}{\lambda^2 - \omega^2} d\lambda$$

We can expand the numerator to $(\lambda^2 - \omega^2) + \omega^2$, separate along the parenthesis into two integrals and observe again the singularity at $\lambda = \omega$ as before. We have as the result

$$\ln \frac{h(\omega)}{h(0)} = \frac{2}{\pi} \Omega_c t_d \left(1 - \frac{\omega}{2\Omega_c} \ln \frac{\Omega_c + \omega}{\Omega_c - \omega} \right) \quad \omega < \Omega_c \quad (97)$$

If we approximate for $\omega \ll \Omega_c$ as in (94), we find a nearly parabolic change of the attenuation

$$\ln \frac{h(\omega)}{h(0)} \approx \frac{2}{\pi} \Omega_c t_d \left[1 - \left(\frac{\omega}{\Omega_c} \right)^2 \right] \quad \omega \ll \Omega_c$$

demonstrating that $h(\omega)$ is an even function in ω . As $\omega \rightarrow \Omega_c$, we have again a logarithmic infinity of the attenuation function, actually with negative sign as (97) indicates. For $\omega > \Omega_c$ we can integrate directly as in (96) because $0 < \lambda < \Omega_c$, so that

$$\ln \frac{h(\omega)}{h(0)} = \frac{2}{\pi} \Omega_c t_d \left(1 - \frac{\omega}{2\Omega_c} \ln \frac{\omega + \Omega_c}{\omega - \Omega_c} \right) \quad \omega > \Omega_c \quad (98)$$

* M. J. DiToro, "Phase and Amplitude Distortion in Linear Networks," *Proc. I.R.E.*, **36**, 24-36 (1948).

† H. Nyquist, "Certain Topics in Telegraph Transmission Theory," *Trans. AIEE*, **47**, 617-644 (1928).

‡ E. D. Sunde, "Theoretical Fundamentals of Pulse Transmission," *Bell System Tech. J.*, **33**, 721-788, 987-1010 (1954). A. W. Norton, Jr., and H. E. Vaughan, "Transmission of Digital Information over Telephone Circuits," *Ibid.*, **34**, 511-528 (1955).

Outside the pass band the attenuation function decreases from the edge of the band, but reaches again large values as $\omega \rightarrow \infty$. As we let $\Omega_c \rightarrow \infty$ we see from (97) that we approach constant amplitude values because the logarithmic term vanishes. To keep the product $\Omega_c t_d$ constant, we also find again that the delay time must decrease.

Obviously, a large variety of simple shapes of either the real part or the imaginary part (amplitude or phase) of network functions can be assumed and a systematic study made of the compatible conjugate part* as, e.g., in Bode,^{B2} chapter XV. Considerable insight into system behavior is thus obtained. Surely, we must avoid sharp discontinuities in either network characteristic because a logarithmic singularity is produced in the conjugate characteristic.

As another general approach, we might use a Fourier series representation of the transfer function as we used in (62) for the method of "paired echos" and determine the coefficients C_n so as to give the most desirable transient response for a given signal. Suppose we choose a step voltage $V_0 I$, and select for simplicity of demonstration

$$\begin{aligned} A(\omega) &= A_0 \left((1 - \alpha) + \alpha \cos \frac{\pi \omega}{\Omega_c} \right) \\ \Phi(\omega) &= \omega t_d \end{aligned} \quad (99)$$

i.e., ideally linear phase characteristic with a drooping amplitude characteristic where α is a parameter to be so selected that we get minimum overshoot. To use the results (65) directly we rewrite

$$A(\omega) = A_0 \left((1 - \alpha) + \frac{\alpha}{2} e^{j\frac{\pi \omega}{\Omega_c}} + \frac{\alpha}{2} e^{-j\frac{\pi \omega}{\Omega_c}} \right)$$

giving in comparison with (62) the coefficient values

$$C_0 = 1 - \alpha, \quad C_1 = \frac{\alpha}{2}, \quad C_{-1} = \frac{\alpha}{2}$$

This permits at once the composition of the total response (65), namely

$$\frac{v_r(t)}{A_0 V_0} = \frac{1 - \alpha}{2} + \frac{1 - \alpha}{\pi} Si(x) + \frac{\alpha}{4} + \frac{\alpha}{2\pi} Si(x + \pi) + \frac{\alpha}{4} + \frac{\alpha}{2\pi} Si(x - \pi)$$

The constant terms combine to the value $\frac{1}{2}$, independent of α , and x is defined by $x = \Omega_c(t - t_d)$ as in (25). We have already emphasized in section 4.4 and illustrated in Fig. 4.15 that less overshoot

* T. Murakami and M. S. Corrington, "Relation between Amplitude and Phase in Electrical Networks," *RCA Rev.*, **9**, 602-631 (1948).

brings a longer build-up time so that a proper compromise must be selected. A detailed discussion of this example and of others is given in Küpfmüller,^{A10} chapter IV; of particular value is the summary, pp. 113–116, in the form of illustrative graphs. Obviously, if we take more terms in the Fourier series for $A(\omega)$, we have more adjustable parameters available and thus can satisfy more stringent conditions. Having determined the desired $A(\omega)$, we then need to synthesize the network.

Attempts have therefore been made either to catalogue some typical transient responses, e.g., to unit step, of a large variety of simple networks in terms of characteristic parameters,* or to provide charts for this purpose,† or to analyze a given transient response directly in terms of standard response curves, e.g., that given in (29), which can be readily associated with typical spectrum functions.‡

A direct method of approximation§ takes the ideal analytic transfer function (10) in the notation of the complex frequency domain

$$H_0(p) = A_0 e^{-pt_d}$$

where A is the constant amplitude and t_d the constant time shift, and expands into the Taylor series

$$H_0(p) = A_0(1 - pt_d + \frac{1}{2}p^2t_d^2 - \frac{1}{6}p^3t_d^3 + \cdots) \quad (100)$$

The transfer function of any lumped-parameter network with linear, bilateral elements is rational in p and also can be expanded into a Taylor series about $p = 0$, giving

$$H(p) = \frac{N(p)}{D(p)} = \frac{\sum_{(m)} a_m p^m}{\sum_{(n)} b_n p^n} = \sum_{n=0}^{\infty} c_n p^n \quad (101)$$

Comparing coefficients of like powers of p in (100) and (101), a series of equations can be established which involve the branch elements of the network

$$c_n = (-1)^n \frac{A_0}{n!} t_d^n \quad (102)$$

* H. E. Kallman, R. E. Spencer, and C. P. Singer, "Transient Response," *Proc. I.R.E.*, **33**, 169–195 (1945).

† A. V. Bedford and G. L. Fredendall, "Analysis, Synthesis, and Evaluation of the Transient Response of Television Apparatus," *Proc. I.R.E.*, **30**, 440–457 (1942).

‡ H. A. Samulon, "Spectrum Analysis of Transient Response Curves," *Proc. I.R.E.*, **39**, 175–186 (1951).

§ N. Marcuvitz, "Distortionless Correction of Networks," *M.E.E. Thesis* Polytechnic Institute of Brooklyn, New York, 1941.

The selection of the elements so as to satisfy (102) will tend to achieve distortionless correction of $H(p)$. Because even powers of p in (100) contribute to the real part of $H(j\omega)$ and odd powers of p to its imaginary part, we can infer that satisfying (102) successively for $n = 0, 1, 2$ means alternation between amplitude and phase correction.

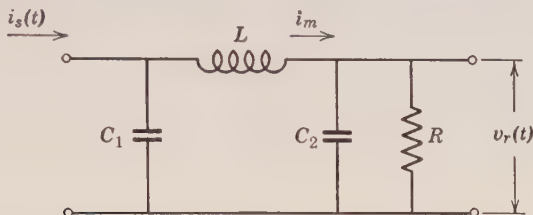


Fig. 4.18. Low-pass filter section as amplifier interstage network.

To take a simple example, we select the interstage low-pass filter section, Fig. 4.18. The transfer impedance is given by

$$Z_T(p) = \frac{V_r(p)}{I_s(p)} = \frac{R}{1 + Rp(C_1 + C_2) + p^2LC_1 + p^3RLC_1C_2}$$

Introducing the simplifying notation

$$q = pR(C_1 + C_2), \quad A = \frac{L}{R^2(C_1 + C_2)}, \quad B = \frac{C_1}{C_1 + C_2}$$

we can write the denominator

$$1 + q + ABq^2 + AB(1 - B)q^3$$

and obtain by long-hand division the expansion

$$Z_T(p) = R[1 - q + (1 - AB)q^2 - (1 - AB - AB^2)q^3 + \cdots] \quad (103)$$

In order to compare this term by term with (100), we introduce there

$$pt_d = q \frac{t_d}{R(C_1 + C_2)} = aq$$

We thus have by comparison of (103) and (100) the relations

$$A_0 = R, \quad a = 1, \quad AB = \frac{1}{2}, \quad AB^2 = \frac{1}{3}$$

or also specifically in terms of the parameters

$$A_0 = R, \quad t_d = R(C_1 + C_2) = 3RC_2$$

$$B = \frac{2}{3}, \quad C_1 = 2C_2$$

$$A = \frac{3}{4}, \quad L = \frac{3}{4}R^2(C_1 + C_2) = \frac{9}{4}R^2C_2$$

If we compare this with the example given in Kallman,* Fig. 13, we find agreement except for the value L which is chosen there slightly differently, namely as $L = 2R^2C_2$. The transient performance of this network is quite satisfactory and is in fact better than that of the so-called shunt-peaking coil stage. Obviously, with more complicated networks more parameters would be available to permit higher order approximation to the ideal case.

The highest order approximation to the ideal transfer function (10) has been obtained with the so-called delay lines.† Actually, the infinitely long simple low-pass wave filter treated in section 3.7 has an ideally flat amplitude characteristic and a phase function nearly linear over at least half the pass band, as shown in Fig. 3.6. Obviously, for higher frequencies the condensers and coils will mutually interact through field linkages and the analysis becomes much more complex, though the mutual inductance effects appear to be beneficial for the performance.‡ The more recent use of helical lines leads into a combination of lumped parameters and transmission-line representation.§

PROBLEMS

4.1 Evaluate the response of an ideal low-pass filter with the characteristics in Fig. 4.3 to a sawtooth pulse voltage $v_s(t) = Vt/T$, $0 < t < T$. Follow the method in section 4.2.

4.2 Assume a narrow-band ideal filter with $h(\omega) = 1$ for $\omega_1 < \omega < \omega_2$, with $\omega_2 - \omega_1 = \Delta\omega$, a small number, and with $\Phi(\omega) = \omega t_d$ in the same range as shown in Fig. 4.12. Apply an impulse voltage $M_v S_0(t)$ and find the response.

4.3 Apply a step voltage to the narrow-band ideal filter of problem 4.2 and find the response. Discuss the dependence of the response upon the center frequency of the filter.

4.4 Two square-wave voltage pulses, each of duration T and spaced an interval T apart, are applied to the low-pass filter in Fig. 4.3. Find the response for various values of cutoff frequency Ω_c . Discuss the resolution of the two pulses as a function of Ω_c .

4.5 Apply to the low-pass filter with the characteristics in Fig. 4.3 a sine wave voltage $\text{Im}(\bar{V}e^{j\omega_0 t})$ with $\omega_0 > \Omega_c$, i.e., outside the pass band. (a) Show that the integration for the inverse Fourier transform can be carried through directly.

* *Op. cit.*, p. 174.

† H. Pender and K. McIlwain, *Electrical Engineers' Handbook, Electric Communication and Electronics*, 4th edition, "Delay Lines," section 9.20 by H. A. Wheeler, Wiley, New York, 1950.

‡ M. J. E. Golay, "The Ideal Low-Pass Filter in the Form of a Dispersionless Lag Line," *Proc. I.R.E.*, **34**, 138P-144P (1946).

§ M. J. DiToro, "General Transmission Theory of Distributed Helical Delay Lines with Bridging Capacitance," *IRE Convention Record, Circuit Theory*, Part 5, 64-70 (1953).

(b) Relate the response to the distance of ω_0 from the pass band. (c) Compare the response with (35) and (38).

4.6 Evaluate the response to a step voltage $V_m 1$ of an ideal low-pass filter with the characteristics shown in Fig. 4.19a. Assume $\omega' = \frac{4}{5} \Omega_c$ and compare the result with Fig. 4.8.

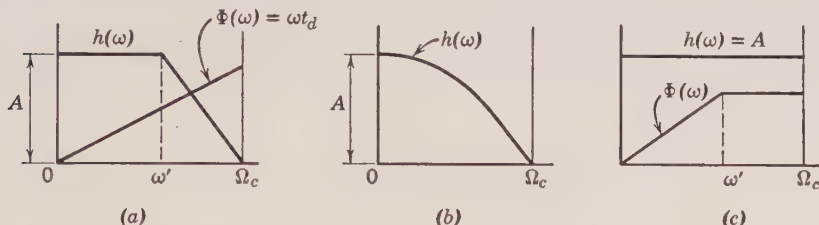


Fig. 4.19. Modified ideal low-pass filter characteristics. (a) Finite amplitude transition to cutoff; (b) cosine amplitude characteristic; (c) constant amplitude but nonideal phase shift.

4.7 Evaluate the response to a step voltage $V_m 1$ of a low-pass filter with the characteristic shown in Fig. 4.19b. The amplitude function is defined as $h(\omega) = A \cos(\pi\omega)/(2\Omega_c)$; the phase shift is assumed to be zero. Compare the result with Fig. 4.8.

4.8 Evaluate the response to a sinusoidal voltage $V_m \sin \omega_0 t$ of a low-pass filter with the idealized amplitude characteristic in Fig. 4.19b (see problem 4.7). (a) Choose $\omega_0 < \Omega_c$, i.e., in the pass band. (b) Choose $\omega_0 > \Omega_c$, i.e., outside the pass band.

4.9 Add to the amplitude characteristic in Fig. 4.19b a linear phase shift for $\omega < \Omega_c$ as in Fig. 4.3. (a) Find the response to a step voltage. (b) Compare with the solution to problem 4.7. (c) Compare with Fig. 4.8.

4.10 A low-pass filter with an ideal constant amplitude characteristic might have a phase shift as shown in Fig. 4.19c defined by $\Phi(\omega) = \omega t_d$ for $\omega < \omega'$, and $\Phi(\omega) = \omega' t_d$ for $\omega' < \omega < \Omega_c$. (a) Apply a step voltage $V_m 1$ and find the response. (b) Compare with Fig. 4.8.

4.11 Take the ideal low-pass filter characteristic of Fig. 4.3 and add to the amplitude only a ripple term for $\omega < \Omega_c$, so that $h(\omega) = A + A_1 \cos(2\pi\omega/\Omega_c)$. (a) Find the response to a step voltage $V_m 1$. (b) Compare the response with Fig. 4.8. (c) Discuss the result as function of A_1 . (d) Draw comparison with real filter characteristic, Fig. 4.5.

4.12 Take the ideal low-pass filter characteristic of Fig. 4.3 and add to the phase characteristic only a ripple for $\omega < \Omega_c$ so that $\Phi(\omega) = \omega t_d + a \sin 2\pi\omega/\Omega_c$. (a) Find the response to a step voltage $V_m 1$. (b) Compare the response with Fig. 4.8. (c) Discuss the result as a function of a .

4.13 Combine the amplitude and phase ripples of problems 4.11 and 4.12 and find the response for a step voltage. (a) Compare the response with the previous results.

4.14 In problem 4.11 the amplitude ripple was selected to cover the positive half bandwidth with one wavelength. Take a ripple of shorter oscillation $\cos(2\pi n\omega/\Omega_c)$ and (a) find the response to a step voltage; (b) discuss the response as a function of the order n .

4.15 In problem 4.12 the phase ripple was selected to cover the positive half bandwidth with one wavelength. Take a ripple of shorter oscillation $\sin(2\pi n\omega/\Omega_c)$ and (a) find the response to a step voltage; (b) discuss the response as a function of the order n .

4.16 Combine the amplitude characteristic from Fig. 4.19a with the phase characteristic from Fig. 4.19c. (a) Find the response to a step voltage. (b) Compare with the previous results.

4.17 Evaluate the compatible phase shift characteristic for the amplitude function in Fig. 4.19a.

4.18 Evaluate the compatible amplitude characteristic for the phase shift characteristic in Fig. 4.19c.

4.19 If a ripple is superimposed upon the amplitude function as in problem 4.11, find the compatible phase shift deviation. This assumes that we have given the compatible phase shift characteristic for the given basic amplitude characteristic.

5. ACTIVE FOURPOLES

Amplifying devices can be conceived as “active” fourpoles possessing an internal energy source which is controlled by the signals applied at either one or both terminal pairs. We differentiate the internal energy source as belonging integrally to the fourpole from the energy sources, usually twopoles, which apply the signals. This differentiation is important to maintain the concept of the passive fourpole as we had defined it in Chapter 2 and to make the active fourpole a complete unit in itself. The most obvious examples of active fourpoles are, of course, electron tubes such as triodes, tetrodes, pentodes, and the transistors; but electromechanical amplifiers, mechanical and hydraulic feed-back control devices or servomechanisms, and many others should also be so considered.

In order to narrow the field to manageable proportions and to stay within the scope of this book, we shall treat only *linear active fourpoles* which permit the application of all the methods of analysis that we have studied for linear passive systems. This will restrict the scope severely in so far as many interesting and new phenomena are caused by nonlinear characteristics. But a large variety of important transient problems are still left that can at least be closely approximated by linear analysis. In the case of electron tubes we shall therefore consider only the relations between the small signal voltages and currents, assuming that sufficient steady biasing voltages and currents are applied to permit the assumption of linear relations throughout.

Again, as in the passive fourpole sections, we shall use the Laplace transform method throughout to obtain our final solutions, though we shall occasionally formulate problems in terms of ordinary differential equations to permit the continuation with any of the other methods.

5.1 Electron Tubes as Active Fourpoles

Among the electron tubes, the triode is the simplest physical structure constituting an active fourpole. Fig. 5.1a shows the conventional arrangement of the three electrodes, each spatially separated from the

other, carried to four terminal posts. We have disregarded the biasing direct voltage, making grid G normally negative, the d-c high-voltage normally applied with the polarity shown to plate (anode) P so as to attract the electron current from cathode K , and also the d-c energy source connected to cathode K to heat it so that it emits electrons. A signal voltage v_g applied at the grid-cathode terminal pair will modulate the electron current i_p flowing from cathode to plate; as electron current its conventionally positive sign is actually from plate to cathode, so that our choice of the direction of i_p in conformity

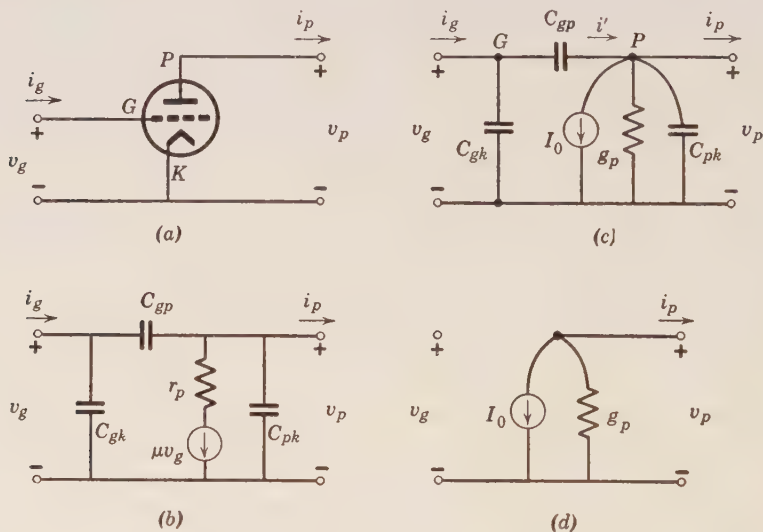


Fig. 5.1. Conventional triode: (a) basic physical structure; (b) equivalent circuit with internal voltage source; (c) equivalent circuit with internal current source; (d) same as (c) but disregarding all capacitances at very low frequencies.

with the general fourpole convention is actually in opposition to physical fact. We must retain it, however, to be consistent, and we shall find negative values for the output voltage v_p as well as the current i_p , which is in keeping with the usually expected "phase reversal" of the signal as it passes through an amplifier stage. [With the opposite choice of output current, see section 2.1, its direction would appear consistent with physical fact; however, in the fourpole equations the authors of the references on p. 53 must write $(-i_p)$ anyway in order to preserve unique expressions for the fourpole parameters.]

To represent the triode in the form of an equivalent circuit* we can

* First done by H. W. Nichols, "The Audion as a Circuit Element," *Phys. Rev.* (2), **13**, 404-414 (1919). Extensive early review in R. W. King, "Thermionic

choose either the mesh or the node point of view. For the former we deduce directly from Fig. 5.1a the circuit elements in Fig. 5.1b, where the obvious interelectrode capacitances form an unsymmetrical passive Π -section, and where the modulated electron stream as internal energy source is represented as the voltage source μv_g with internal resistance r_p , the *plate* resistance. This assumes, of course, simultaneity of electron flow modulation throughout the entire tube, i.e., we disregard transit time effects; it is therefore restricted to frequencies below which electron inertia effects become noticeable.* The *amplification factor* μ can be computed with considerable accuracy from the geometry of the tube; the *plate resistance* r_p is usually found by measurement. The polarity of this internal voltage source must be from plate to cathode in accordance with the physical principles.

If we choose, as definitely more advantageous, the nodal analysis of vacuum-tube circuits, then Fig. 5.1c shows the equivalent internal current source I_0 with shunt admittance $g_p = 1/r_p$. The value of the current is related to the active voltage by the relations

$$r_p I_0 = \mu v_g, \quad I_0 = g_p \mu v_g = g_m v_g \quad (1)$$

where g_m is usually called the *transconductance of the triode*. The passive Π -section of the capacitances remains the same, of course.

For very low frequencies it is convenient and permissible to suppress the capacitances which have very small values in any case. The equivalent circuit simplifies then to Fig. 5.1d, which is, indeed, described completely by the active element, the current source with shunt admittance g_p . Here we have a purely resistive active fourpole and we can write directly for the time functions

$$i_p(t) = -I_0(t) - g_p v_p(t), \quad v_g(t) = v_g(t) \quad (2)$$

If we use (1), then this gives a relation between input and output quantities

$$\dot{i}_p = -g_m v_g - g_p v_p, \quad v_g = v_g \quad (3)$$

from which we can easily obtain the fourpole parameters.

Introducing the standard fourpole notation, in which

$$i_g = i_1, \quad v_g = v_1; \quad i_p = i_2, \quad v_p = v_2 \quad (4)$$

Vacuum Tubes," *Bell System Tech. J.*, **2**, 31-100 (1923). See also any modern textbook on electronics or radio engineering.

* F. B. Llewellyn, *Electron Inertia Effects*, Cambridge University Press, England, 1943; W. R. Ferris, "Input Resistance of Vacuum Tubes as Ultra-high-frequency Amplifiers," *Proc. I.R.E.*, **24**, 82-107 (1936).

we can write (3) in matrix form

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = - \begin{bmatrix} 1/\mu & 1/g_m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ i_2 \end{bmatrix}$$

and thus state the standard parameter representation of the ideal triode at low frequencies

$$\alpha = -1/\mu, \quad \beta = -1/g_m, \quad \mathcal{C} = \mathcal{D} = 0 \quad (5)$$

This is obviously a nonbilateral network; in fact, we cannot invert the parameter matrix; we cannot solve for i_2, v_2 in terms of the input values i_1, v_1 . Such a matrix is called a *singular matrix*. We also observe that its determinant

$$\alpha\mathcal{D} - \beta\mathcal{C} = 0 \quad (6)$$

whereas for the passive fourpole we had unity value for the determinant as demonstrated in (2.19).

If we apply the nodal analysis to the complete equivalent circuit of the triode in Fig. 5.1c, we have for the two node pairs at P and G , respectively

$$\begin{aligned} i' &= I_0 + i_p + g_p v_p + C_{pk} \frac{dv_p}{dt} \\ i_g &= i' + C_{gk} \frac{dv_g}{dt} \end{aligned} \quad (7)$$

The linking current i' can be expressed in terms of the node-pair voltage difference

$$i' = C_{gp} \frac{d}{dt} (v_g - v_p) \quad (8)$$

and can of course be eliminated from (7). We thus have two equations for the four related quantities v_g, i_g, v_p, i_p , and if we can define the terminal conditions at each terminal pair the system can be solved readily for any applied signal voltage v_g by whichever method we please to choose.

For the general fourpole parameter representation, we must, however, choose either steady-state a-c conditions so that the indicated derivative operations can be carried out in general form, or use operational or transform methods in order to convert the system of differential equations in (7) into an algebraic system. The latter is preferable because we can use it for steady state as well, if we replace $p = j\omega$. In Laplace transform notation we can now rewrite (7) as follows, if we

make the assumption of a de-energized initial state and also replace i' by (8)

$$\begin{aligned} pC_{gp}(V_g - V_p) &= g_m V_g + I_p + (g_p + pC_{pk})V_p \\ I_g &= p(C_{gp} + C_{gk})V_g - pC_{gp}V_p \end{aligned} \quad (9)$$

Let us introduce again for the Laplace transforms the standard indices

$$V_g = V_1, \quad I_g = I_1; \quad V_p = V_2, \quad I_p = I_2$$

as in (4) so that we have the standard fourpole notation, and let us solve (9) for these two pairs of terminal quantities. We find, repeating the standard terminology first

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad (10)$$

that

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{\text{Triode}} = \frac{1}{y} \begin{bmatrix} Y_p & 1 \\ Y_p Y_g - ypC_{gp} & Y_g \end{bmatrix} \quad (11)$$

where

$$\begin{aligned} Y_p &= g_p + p(C_{pg} + C_{pk}) \\ Y_g &= p(C_{gp} + C_{gk}) \\ y &= -\mu g_p + pC_{pg} \end{aligned} \quad (12)$$

The quantities Y_p and Y_g have simple physical significance; they are the sum of all admittance elements joining at nodes P and G respectively, whereas y is significant for the active *transadmittance* between grid and plate. If we let all the capacitances go to zero, we recover at once the simplified matrix with elements (5). This interpretation will permit quick generalization if we desire to combine some external circuit elements with the tube parameters in order to simplify expressions. Thus, we could account for any grid-input resistance R_g placed between grid and cathode by adding $G_g = 1/R_g$ into Y_g in (12); or for any load impedance $Z_L(p)$ by adding $Y_L(p) = 1/Z_L(p)$ into Y_p .

The determinant of the general fourpole parameters is here

$$\mathfrak{A}\mathfrak{D} - \mathfrak{B}\mathfrak{C} = \frac{pC_{pg}}{y} = \frac{pC_{pg}}{pC_{pg} - \mu g_p} = \eta \quad (13)$$

neither unity nor zero; we shall designate it by η . If we disregard the grid-plate capacitance, we again obtain zero, as in (6); it should be observed that it is the link between grid and plate which decisively affects the value of this determinant. If we let the tube be cold, so

that only the passive Π -section of capacitances remains and $\mu = 0$, then (13) reduces to unity as it should for passive circuits.

We can reiterate the nonbilateral nature of the triode by expressing the current-voltage relations in terms of the impedance matrix as in (2.14) for the passive fourpole, and, as there, taking the output current with negative sign by solving directly for the voltages in terms of the currents we get from (9)

$$\begin{bmatrix} V_1(p) \\ V_2(p) \end{bmatrix} = (Y_p Y_g - ypC_{gp})^{-1} \begin{bmatrix} Y_p & pC_{gp} \\ pC_{gp} & \mu g_p Y_g \end{bmatrix} \begin{bmatrix} I_1(p) \\ -I_2(p) \end{bmatrix} \quad (14)$$

The terms of the admittance matrix are the numerators of the impedances $Z_{11}(p)$, $Z_{12}(p)$, $Z_{21}(p)$, and $Z_{22}(p)$, respectively. It is interesting to note that the difference

$$Z_{12}(p) - Z_{21}(p) = \mu g_p = g_m \quad (15)$$

is exactly the active transconductance of the triode, going to zero if $\mu = 0$, i.e., if the tube is taken cold. We also observe by comparison with (2.19b) that

$$\eta = \frac{Z_{12}}{Z_{21}} \quad (15a)$$

i.e., that the nonbilateral nature of the active fourpole is directly responsible for $\eta \neq 1$. Perhaps we should note here that the active fourpole is always nonreciprocal or nonbilateral, whereas the reverse need not be true. The gyrator is a passive nonbilateral element with $\eta = -1$ as demonstrated in (2.19c).

The treatment of the triode by means of matrices was first reported* in 1930, but only recently, and then only in the simpler cases, has the use of matrix notation become more widespread† though by far not as frequent as the great convenience and possibility of systematic treatment would indicate. Perhaps a contributing factor is the loss of the physical concepts in the forest of mathematical developments.

* A. C. Bartlett, "Multi-stage Valve Amplifier," *Phil. Mag.* (7), **10**, 734-738 (1930). F. Strecker and K. Feldtkeller, "Theory of Low Frequency Amplifier Chains" [German], *Arch. Elektrotech.*, **24**, 425-468 (1930). See also Feldtkeller,^{A6} p. 150.

† W. R. Abbott, "Analysis of Four-Terminal Networks Containing Vacuum Tubes," Misc. Paper 46-204, *AIEE*, Sept. 1946; J. S. Brown and F. D. Bennett, "The Application of Matrices to Vacuum-Tube Circuits," *Proc. I.R.E.*, **36**, 844-852 (1948); H. Epstein, "Solution of Transients in Active Four-Terminal Networks," *J. Franklin Inst.*, **251**, 607-616 (1951); H. Hsu, "On Transformations of Linear Active Networks with Applications at Ultra-High Frequencies," *Proc. I.R.E.*, **41**, 59-67 (1953).

The equivalent active fourpole for the *pentode* is very easily obtained. Fig. 5.2*a* shows the physical structure of the pentode with control grid G_1 , normally carrying the input signal, shielded by screen grid G_2 and suppressor grid G_3 from plate P . This leads to (almost) complete independence of the plate current i_p from the plate voltage v_p , since the capacitance $C_{gp} = 0$; in turn, (13) shows that the determinant of the fourpole parameters vanishes in this case. The circuit of Fig. 5.2*b* is practically identical with Fig. 5.1*c* with the omission of C_{gp} , so that we

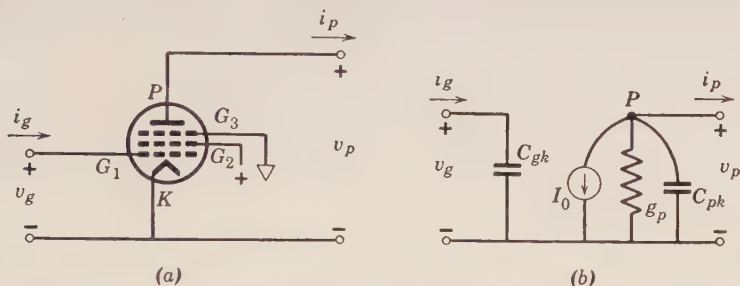


Fig. 5.2. Conventional pentode: (a) basic physical structure; (b) equivalent circuit for node analysis.

can take all the above forms for the conventional triode and duplicate them here. We have

$$\begin{aligned} Y_p &= g_p + pC_{pk} \\ Y_g &= pC_{gk} \end{aligned} \quad (16)$$

$$y = -\mu g_p = -g_m$$

and correspondingly

$$\begin{bmatrix} \alpha & \beta \\ \epsilon & \mathfrak{D} \end{bmatrix}_{\text{Pentode}} = -\frac{1}{g_m} \begin{bmatrix} Y_p & 1 \\ Y_p Y_g & Y_g \end{bmatrix} \quad (17)$$

Although the definition of Y_p in (16) and its representation in Fig. 5.2*b* indicates it as capacitive reactance between plate and cathode, we must define it to include the capacitance between plate and suppressor grid G_3 which normally is at ground potential and usually connected directly to the cathode. The actual value of C_{pk} will therefore be considerably larger than in triodes.

Last, we shall consider the grounded-plate triode or “*cathode follower*” with the basic physical arrangement shown in Fig. 5.3*a*. Obviously the electron current must still go from cathode to plate, or, in conventional notation, the current source must have polarity from plate to cathode so that I_0 in Fig. 5.3*b* now is oppositely directed with

respect to the output voltage v_2 so that no "phase reversal" is to be expected as in the conventional triode. Indeed, this is one of the characteristics of the cathode follower tube when used in an amplifier circuit (see section 5.2). It is important to recall that we are dealing only with the time variable components of currents and voltages and are disregarding completely the comparatively much larger d-c components; yet, the plate must carry the normal positive d-c potential difference against the cathode to attract the electrons and the grid will be properly biased with respect to the cathode in order to assure adequate control action by the signal voltage. Actually, the effective

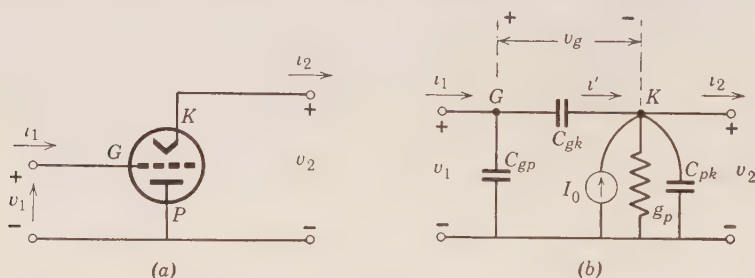


Fig. 5.3. "Cathode follower" connection or "grounded-plate" triode: (a) physical arrangement; (b) equivalent circuit for node analysis.

grid voltage as far as (1), the basic active element relation is concerned, v_g is now the voltage between nodes G and K in Fig. 5.3b. Applying the nodal analysis to the complete circuit, as we did in (7) for the conventional triode, gives

$$\begin{aligned} i' &= -I_0 + i_2 + g_p v_2 + C_{pk} \frac{dv_2}{dt} \\ i_1 &= i' + C_{gp} \frac{dv_1}{dt} \end{aligned} \quad (18)$$

The linking current can again be expressed in terms of the node-pair voltage difference

$$i' = C_{gk} \frac{d}{dt} (v_1 - v_2) \quad (19)$$

and we also have from (1) with the grid voltage as already stated

$$I_0 = g_m v_g = g_m (v_1 - v_2) \quad (20)$$

Eliminating both I_0 and i' from (18), we obtain two differential equations for the four related quantities v_1 , i_1 , v_2 , i_2 , and if we define the

terminal conditions at each terminal pair the system can be solved readily for any applied voltage by any one of the available methods of analysis.

Again, we prefer to use the Laplace transform method as permitting the most general utilization. Assuming the de-energized initial state—any initial charges and currents can be treated independently by the superposition principle—we can rewrite (18) with the substitutions (19) and (20) in Laplace transforms as follows

$$pC_{gk}(V_1 - V_2) = -\mu g_p(V_1 - V_2) + I_2 + (g_p + pC_{pk})V_2$$

$$I_1 = pC_{gk}(V_1 - V_2) + pC_{gp}V_1$$

Solving for the two pairs of terminal quantities in the standard form (10), we find here, similar to (11)

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{\text{Cath F.}} = \frac{1}{y} \begin{bmatrix} Y_k & 1 \\ Y_k Y_g - ypC_{gk} & Y_g \end{bmatrix} \quad (21)$$

where now

$$Y_k = (\mu + 1)g_p + p(C_{kg} + C_{kp})$$

$$Y_g = p(C_{gk} + C_{gp}) \quad (22)$$

$$y = \mu g_p + pC_{gk}$$

The quantities Y_k and Y_g have relatively the same physical significance as for the conventional triode; they represent the sum total of all admittance elements joining at nodes K and G of Fig. 5.3b; y is significant for the active *transadmittance* between grid and cathode. Again, this will permit quick generalization if we desire to combine external circuit elements with the tube parameters for specific applications. Thus, we can account for any input impedance Z_i paralleling C_{gp} by simply adding $Y_i = 1/Z_i$ into Y_g . If we do this and simultaneously let all capacitances go to zero, (21) simplifies to

$$\begin{bmatrix} \frac{\mu + 1}{\mu} & \frac{1}{g_m} \\ \frac{\mu + 1}{\mu Z_i} & \frac{1}{g_m Z_i} \end{bmatrix}$$

the form given in Brown-Bennett* as (7c).

The determinant of the general fourpole parameters is here again

$$\mathfrak{A}\mathfrak{D} - \mathfrak{B}\mathfrak{C} = \frac{pC_{gk}}{y} = \frac{pC_{gk}}{pC_{gk} + \mu g_p} = \eta \quad (23)$$

* *Op. cit.*, *Proc. I.R.E.*, p. 846.

going to zero if we disregard the grid-cathode capacitance, and becoming unity if we take the tube as cold so that $\mu \rightarrow 0$.

It should be obvious now how we can apply the matrix method to obtain the characteristic active fourpole parameters of any tube with any physical arrangement, as, e.g., the grounded-grid triode, for which

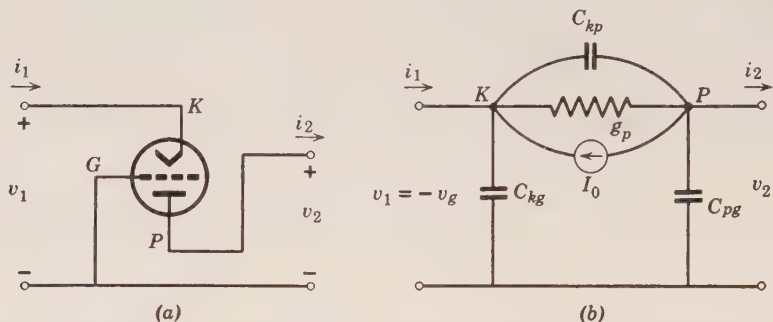


Fig. 5.4. Grounded-grid triode: (a) physical arrangement; (b) equivalent circuit for node analysis.

Fig. 5.4b gives the equivalent circuit. Without carrying through the detail, the active fourpole parameters are found as

$$\begin{bmatrix} \alpha & \beta \\ \mathcal{C} & \mathcal{D} \end{bmatrix}_{\text{Gr. Grid}} = \frac{1}{y} \begin{bmatrix} Y_p & 1 \\ Y_p Y_k - y(g_p + pC_{pk}) & Y_k \end{bmatrix} \quad (24)$$

where now

$$\begin{aligned} Y_p &= g_p + p(C_{pk} + C_{pg}) \\ Y_k &= (\mu + 1)g_p + p(C_{kp} + C_{kg}) \\ y &= (\mu + 1)g_p + pC_{pk} \end{aligned} \quad (25)$$

5.2 Transients in Simple Amplifiers

Although the performance of amplifiers has been generally studied for steady-state a-c signals, the transient analysis has not kept pace, primarily because of the considerable complexity of the relations with which we are confronted if we do not employ a systematic approach. We have observed that all the tube matrices are of the form

$$\begin{bmatrix} \alpha & \beta \\ \mathcal{C} & \mathcal{D} \end{bmatrix}_{\text{Tube}} = \frac{1}{y} \begin{bmatrix} a & 1 \\ ad - y^2\eta & d \end{bmatrix} \quad (26)$$

where y , a , and d are admittances and can be taken from the respective detail forms for each particular tube arrangement, and η is the value

of the determinant as in (13)

$$\alpha\mathfrak{D} - \beta\mathfrak{C} = \eta$$

and can also be read off the individual matrices; $\eta = 1$ for $\mu = 0$, i.e., for the cold tube which reduces to a passive network.

If the tube is connected to a passive network which from the node-pair point of view can be interpreted as the load admittance Y_L whatever its actual composition, we can take it as the cascade connection of an active and passive fourpole as shown schematically in Fig. 5.5 with the over-all terminal quantities V_i , I_i for the input, and V_o , I_o

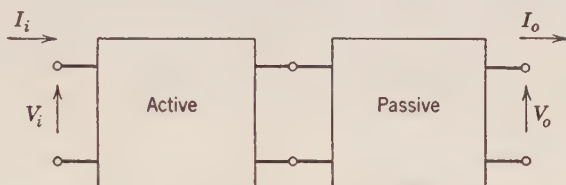


Fig. 5.5. Cascade connection of active fourpole (electron tube) and passive fourpole.

for the output. The over-all parameters are obtained by matrix multiplication

$$\begin{aligned} \begin{bmatrix} V_i \\ I_i \end{bmatrix} &= \begin{bmatrix} \alpha & \beta \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{\text{Tube}} \begin{bmatrix} 1 & 0 \\ Y_L & 1 \end{bmatrix}_{\text{Load}} \begin{bmatrix} V_o \\ I_o \end{bmatrix} \\ &= \frac{1}{y} \begin{bmatrix} a + Y_L & 1 \\ ad - y\eta^2 + dY_L & d \end{bmatrix} \begin{bmatrix} V_o \\ I_o \end{bmatrix} \quad (27) \end{aligned}$$

Here we used (26) for the tube matrix and (2.22) for the load admittance without any restrictive assumptions. If we cascade several identical stages of amplification, we need only multiply the total matrix by itself a number of times equal to the number of stages. Generally we are interested in the output voltage only, assuming the input impedance to the succeeding fourpole as infinitely high. Should this not be warranted, then Y_L could incorporate this additional input impedance.

For the single stage of amplification then, with Y_L as load admittance, the output voltage V_o is to be considered as the open-circuit voltage so that $I_o = 0$. For this condition (27) gives as the relation between input and output voltages the very simple form

$$V_i = (a + Y_L) \frac{V_o}{y}$$

or also

$$V_o = \frac{y}{a + Y_L} V_i \quad (28)$$

For steady-state a-c signals the voltage symbols should be those of the phasors and the admittances are the normal a-c values with $p = j\omega$. For example, the conventional triode has from (11) and (12)

$$a = Y_p = g_p + p(C_{pg} + C_{pk}), \quad y = -\mu g_p + pC_{pg}$$

so that the *voltage amplification* in complex form follows as a function of frequency

$$\frac{V_o}{V_i} = \frac{-\mu g_p + j\omega C_{pg}}{g_p + j\omega(C_{pg} + C_{pk}) + Y_L} \quad (29)$$

If the load is a resistance $R_L = 1/G_L$, and if we disregard all the inter-electrode capacitances of the triode at low frequencies, we obtain

$$\frac{V_o}{V_i} = \frac{-g_m}{g_p + G_L} \quad (30)$$

and finally, if the load admittance G_L is negligibly small compared with $g_p = 1/r_p$, (30) reduces to the negative amplification factor $-\mu$ in accordance with the phase reversal in the conventional pure resistive amplifier stage.

If we desire the transient response to a step voltage $V_m 1$ applied at the input terminals of the de-energized amplifier, we must take (28) as the relation between the Laplace transforms of input and output voltages. For the input step voltage we have $V_i(p) = V_m/p$, and if we assume the load to be resistive, $Y_L = G_L$, then (28) gives for the output voltage transform

$$V_o(p) = \frac{-\mu g_p + pC_{pg}}{p[G + pC_p]} V_m \quad (31)$$

where $G = g_p + G_L$ and $C_p = C_{pk} + C_{pg}$ are the total conductance and total capacitance respectively, at the plate node P , as indicated in Fig. 5.6. Let us define the damping coefficients (inverse time constants)

$$\delta = \frac{G}{C_p} = \frac{g_p + G_L}{C_{pg} + C_{pk}}, \quad \delta' = \frac{g_m}{C_{pg}} \quad (32)$$

then (31) can be written

$$V_o(p) = \frac{C_{pg}}{C_p} \frac{p - \delta'}{p(p + \delta)} V_m$$

and either by expansion into partial fractions and use of Table 1.3 or by the use of the expansion theorem, line 12 of Table 1.4, we find as the inverse Laplace transform

$$v_o(t) = \left[-\frac{g_m}{G} I + \left(\frac{C_{pg}}{C_p} + \frac{g_m}{G} \right) e^{-\delta t} \right] V_m \quad (33)$$

At time $t = 0$ the voltage rises abruptly to the value $(C_{pg}/C_p)V_m$, usually a small fraction of the step amplitude V_m and then decreases

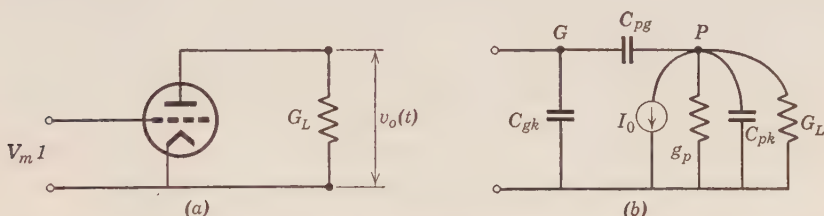


Fig. 5.6. Conventional triode with resistance load and applied step voltage: (a) physical arrangement; (b) equivalent circuit for node analysis.

exponentially to the final negative value (phase reversal)

$$\frac{g_m}{G} V_m = \frac{\mu g_p}{g_p + G_L} V_m$$

For applications it is important to consider the time constant of the exponential approach to steady state because it will define the resolution of individual pulses of rectangular shape. This time constant is

$$T = \frac{1}{\delta} = \frac{C_{pg} + C_{pk}}{g_p + G_L}$$

To make it small requires small interelectrode capacitances and large values of plate and load conductances. Since the former are of the order of 10^{-10} to 10^{-11} farad, and the plate conductance of the order of 10^{-4} mho, the time constant will generally be of the order of 0.1 to 1 microsecond. Any additional resistive or capacitive circuit elements added at plate node P will increase this time constant.

Actual improvement of the build-up time can be secured by the addition of an inductance so as to produce oscillatory response without, however, letting the oscillation predominate. An amplifier with such a circuit is frequently referred to as a *compensated* amplifier, and the arrangements of inductance in shunt as in Fig. 5.7*a*, or in series, as in

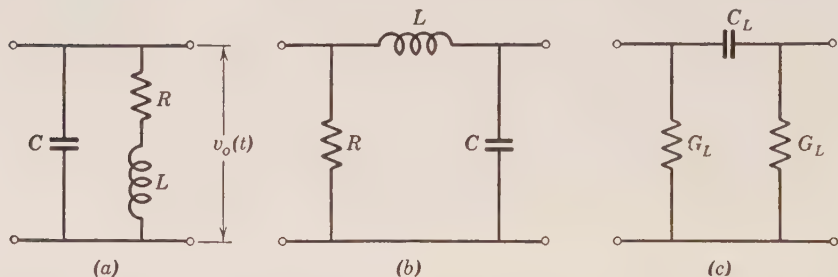


Fig. 5.7. Passive coupling circuits for amplifiers: (a) shunt-peaked compensation; (b) series-peaked compensation, (c) capacitive coupling.

Fig. 5.7*b*, *shunt-peaked* and *series-peaked* compensation, respectively.* The load admittance of the shunt-peaked circuit is given by

$$Y_L = pC + \frac{1}{R + pL} = \frac{p^2 + \delta''p + \Omega_0^2}{p + \delta''} C$$

with $\delta'' = R/L$, $\Omega_0^2 = 1/LC$. Let us choose here a *pentode amplifier*; then (16) and (17) define the tube parameters. Introducing these and $1/L$ into (28) leads to

$$V_o(p) = \frac{V_m}{p} \frac{-g_m(p + \delta'')}{(p + \delta'')(g_p + pC_{pk}) + C(p^2 + \delta''p + \Omega_0^2)} \quad (34)$$

where we have again assumed a step voltage $V_m 1$ applied at the input terminals. If we take the factor $(C + C_{pk}) = C_p$ outside the denominator, we can write it

$$p^2 + (\delta' + \delta'')p + \left(\delta'\delta'' + \frac{C}{C_p} \Omega_0^2 \right)$$

with $\delta' = g_p/C_p$. The roots of this denominator are easily found. It will, however, simplify matters considerably if we also assume the

* See excellent oscillograms of response in G. B. Hoadley and W. A. Lynch, "Transients in Coupling Circuits," *Communications*, **22**, 32-38 (June); 22-28 (July 1943).

two damping coefficients δ' and δ'' to be equal, namely

$$\delta' = \delta'' = \frac{R}{L} = \frac{g_p}{C + C_{pk}} \equiv \delta \quad (35)$$

because now the roots are

$$p_{1,2} = -\delta \pm j\Omega, \quad \Omega = \Omega_0 \sqrt{\frac{C}{C + C_{pk}}} = \frac{1}{\sqrt{L(C + C_{pk})}} \quad (35a)$$

Applying the expansion theorem, line 12 from Table 1.4, we find for the final solution

$$v_o(t) = \mu V_m \left(-\frac{R}{R + R_p} 1 + \frac{\delta}{\sqrt{\delta^2 + \Omega^2}} e^{-\delta t} \cos(\Omega t + \Phi) \right) \quad (36)$$

$$\Phi = \tan^{-1} \frac{\Omega}{\delta}$$

We can simplify this further by introducing

$$\frac{\Omega}{\delta} = Q = \frac{\Omega L}{R} = \frac{1}{R} \sqrt{\frac{L}{C + C_{pk}}} = \sqrt{\frac{r_p}{R}}$$

where the last form follows from (35). This gives now

$$\frac{\delta}{\sqrt{\delta^2 + \Omega^2}} = \sqrt{\frac{R}{R + r_p}} = \sqrt{A}$$

where A is used for abbreviation. Thus, (36) becomes

$$v_o(t) = -\mu V_m A \left(1 - \frac{e^{-\delta t}}{\sqrt{A}} \cos(\Omega t + \Phi) \right) \quad (36a)$$

with

$$\delta = \Omega/Q, \quad \Phi = \tan^{-1} Q, \quad \cos \Phi = \sqrt{A}$$

We see that $v_o(0) = 0$, which we could have ascertained directly from (34) in accordance with line 1b of Table 1.4

$$\lim_{p \rightarrow \infty} pV_o(p) = 0 = v_o(0^+)$$

For the value $Q = 0.5$ Fig. 5.8 shows the actual response to the step voltage. For convenience, the scale of negative values has been plotted upwards and we used

$$\Omega t = x, \quad \tan \Phi = 0.5, \quad \delta t = \frac{x}{Q} = 2x, \quad \sqrt{A} = 0.8944$$

Obviously, the response shows only a slight overshoot and rather rapid rise. The actual rate of rise is defined essentially by the frequency $\Omega/2\pi$ since one quarter period will be the approximate time for $\cos \Omega t$ to decrease to zero value. Obviously, the damping coefficient δ must be large enough to prevent noticeable oscillations. Other circuits have been proposed and various criteria for the evaluation of the transient performance have been set, particularly for amplifiers used in television systems.*

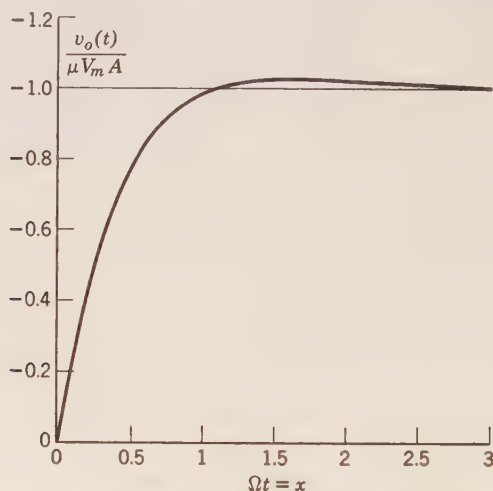


Fig. 5.8. Response of shunt-peaked compensated amplifier to step voltage.

So far we have considered illustrations of loads that can be defined as admittances so that the very simple relation (28) was valid. If we place a general passive fourpole at the output terminals of the tube as in Fig. 5.5, then instead of (27) we have to form the more general matrix product

$$\begin{bmatrix} V_i \\ I_i \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{\text{Tube}} \begin{bmatrix} \alpha & \beta \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{\text{Load}} = \begin{bmatrix} V_o \\ I_o \end{bmatrix} \quad (37)$$

Introducing the tube matrix from (26) and designating the parameters of the load with the subscript L , we can carry through the multiplication readily enough. However, we may again take the output ter-

* R. C. Palmer and L. Mautner, "A New Figure of Merit for the Transient Response of Video Amplifiers," *Proc. I.R.E.*, **37**, 1073-1077 (1949). P. R. Aigrain and E. M. Williams, "Design of Optimum Transient Response Amplifiers," *Ibid.*, 873-879.

minal pair as open-circuited so that $I_o = 0$, and we deduce from (37) the simpler form

$$V_i(p) = (a\mathfrak{G}_L + \mathfrak{C}_L) \frac{V_o(p)}{y}$$

or similar to (28), but more general

$$V_o(p) = \frac{y}{a\mathfrak{G}_L + \mathfrak{C}_L} V_i(p) \quad (38)$$

Depending upon the structure of the passive fourpole, this can readily be interpreted for steady-state a-c signals or for the evaluation of transients.

Let us apply it to the same triode as in Fig. 5.6*a* but terminated into the symmetrical Π -section of Fig. 5.7*c*. With the aid of (2.24) we find

$$\mathfrak{G}_L = 1 + \frac{G_L}{pC_L}, \quad \mathfrak{C}_L = G \left(2 + \frac{G_L}{pC_L} \right)$$

We take a and y as above; for the step voltage $V_m t$ applied to the input terminals we have the Laplace transform V_m/p . Thus, it follows from (38)

$$V_o(p) = \frac{V_m}{p} \frac{pC_L(-g_m + pC_{pg})}{(g_p + pC_p)(G_L + pC_L) + G_L(G_L + 2pC_L)} \quad (39)$$

Using the same time constants as in (32), leaving G_L to stand for the first shunt conductance in the Π -section, we can rewrite (39) in the better form

$$V_o(p) = \frac{C_{pg}}{C_p} \frac{p - \delta'}{p^2 + 2\alpha p + \gamma^2} V_m$$

with

$$2\alpha = \delta + \delta_L + \delta'', \quad \gamma^2 = \delta\delta_L$$

and

$$\delta_L = \frac{G_L}{C_L}, \quad \delta'' = \frac{G_L}{C_p}$$

The inverse Laplace transform can again be found either by partial fraction expansion or by application of the expansion theorem from Table 1.4. The roots of the denominator are

$$p_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \gamma^2}$$

and will always be real as examination of α^2 and γ^2 demonstrates. We thus obtain the final solution

$$v_o(t) = -\frac{V_m}{2\sqrt{\alpha^2 - \gamma^2}} \frac{C_{pg}}{C_p} [(\alpha - \delta' - \sqrt{\alpha^2 - \gamma^2})e^{p_1 t} - (\alpha - \delta' + \sqrt{\alpha^2 - \gamma^2})e^{p_2 t}] \quad (40)$$

Obviously, the time constant of the second exponential is very short

$$T_2 = -\frac{1}{p_2} = \frac{1}{\alpha + \sqrt{\alpha^2 - \gamma^2}}$$

and its amplitude is by far the larger. The response will therefore be a very quick decrease to a rather low level and then a slow further decrease to zero value eventually. If the unit step is indicative of a discharge pulse, we can take T_2 as the measure of the possible repetition rate.

Numerous other examples could as easily be treated with the usual major difficulty residing in the evaluation of the natural modes of response. As a final illustration consider a very simple example of a cathode follower (grounded-plate triode) to which a linearly increasing voltage $v_i(t) = (t/T)V$ is applied. As an amplifier for television purposes it might be terminated into an admittance $Y_L = (G_L + pC_L)$ as shown in Fig. 5.9a. The active fourpole parameters for the tube itself are given in (21) and (22) and the response or output voltage transform has been derived in (28). If we disregard all interelectrode capacitances, the special values of the needed tube parameters are

$$a = Y_k = (\mu + 1)g_p, \quad y = \mu g_p = g_m$$

The transform of the applied voltage is from Table 1.3

$$\mathfrak{L}v_i(t) = \frac{V}{T} \mathfrak{L}t = \frac{V}{Tp^2}$$

so that (28) takes the explicit form

$$V_o(p) = \frac{g_m}{(\mu + 1)g_p + G_L + pC_L} \frac{V}{Tp^2} \quad (41)$$

Introducing the damping coefficient $\delta = (1/C_L)[(\mu + 1)g_p + G_L]$, we get the simple form and solution

$$v_o(t) = \frac{g_m V}{C_L T} \mathfrak{L}^{-1} \frac{1}{p^2(p + \delta)} = \frac{g_m V}{C_L T} \frac{e^{-\delta t} + \delta t - 1}{\delta^2} \quad (42)$$

For very small values of time we can expand the exponential function

$$e^{-\delta t} = 1 - \delta t + \frac{(\delta t)^2}{2} - \dots$$

so that

$$v_o(t) = \frac{g_m V}{C_L T} \cdot \frac{t^2}{2} \quad \delta t \ll 1$$

whereas after a very long time we can assume $e^{-\delta t} \rightarrow 0$, so that

$$v_o(t) = \frac{g_m}{(\mu + 1)g_p + G_L} \left(\frac{t}{T} V - \frac{C_L}{(\mu + 1)g_p + G_L} \frac{V}{T} \right) \quad \delta t \gg 1$$

The total response is as shown in Fig. 5.9c; the output voltage increases first parabolically and then becomes a true replica of the

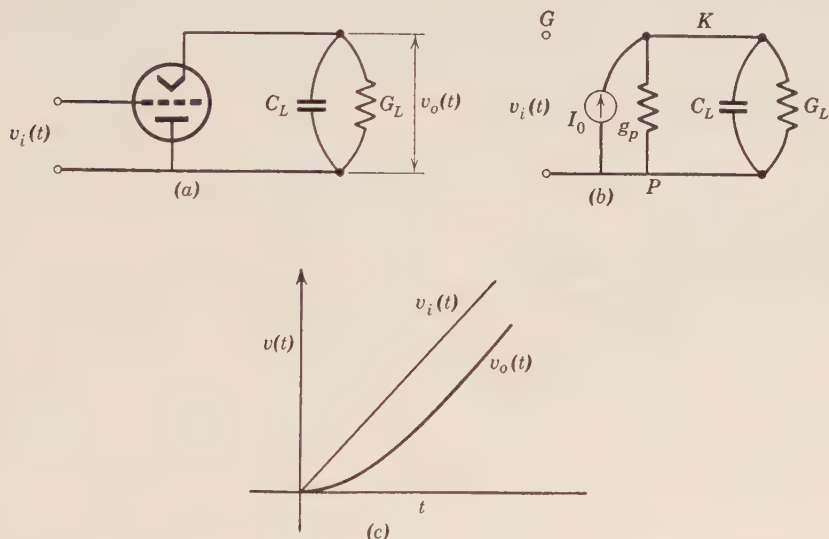


Fig. 5.9. Response of cathode follower to linearly rising voltage: (a) physical arrangement; (b) equivalent circuit; (c) time response.

applied voltage, but “delayed” by the time $\tau = 1/\delta$; we might interpret this also as a “loss of voltage” in so far as only a fixed total time is allowed for the build-up.*

5.3 Transients in Amplifier Chains

If the input signal is weak, several stages of amplification might be necessary to provide a sufficiently strong output signal. The ampli-

* B. Y. Mills, “Transient Response of Cathode Followers in Video Circuits,” *Proc. I.R.E.*, **37**, 631–633 (1949).

fication obtainable in any one stage will depend upon the circuit. However, it is readily demonstrated that the product of useful bandwidth and either power gain or voltage or current amplification factors remains fairly fixed. We can illustrate this, e.g., for the simple triode amplifier with resistive load. The complex voltage amplification is given by (29) as

$$A_v = \frac{V_o}{V_i} = \frac{-g_m + j\omega C_{pg}}{(g_p + G_L) + j\omega(C_{pg} + C_{pk})} \quad (43)$$

The amplification factor at $\omega = 0$ is the same as given by (30)

$$A_v(0) = \frac{-g_m}{g_p + G_L}$$

which is the largest value of (43). The definition of "amplitude bandwidth" is rather arbitrary. We might choose here that value $\omega = \omega_c$ at which the absolute value of (43) is one-half of the amplification factor at $\omega = 0$, or

$$\frac{g_m^2 + \omega_c^2 C_{pg}^2}{G^2 + \omega_c^2 C^2} = \frac{1}{4} \frac{g_m^2}{G^2} \quad (44)$$

where we abbreviated

$$G = g_p + G_L, \quad C = C_{pg} + C_{pk} \quad (44a)$$

This leads to

$$\omega_c^2 = \frac{3g_m^2 G^2}{g_m^2 C^2 - 4G^2 C_{pg}^2} \approx 3 \frac{G^2}{C^2}$$

and thus to the product value "amplification \times bandwidth"

$$|A_v(0)|\omega_c = \frac{g_m}{G} \omega_c \approx \frac{g_m}{C} \sqrt{3}$$

which is clearly independent of the load. For wide-band amplification we would thus have to accept less amplification per stage.

Since the definition of "bandwidth" is rather loose, many different proposals have been made. A simple definition has been proposed by W. W. Hansen*

$$\omega_c = \frac{1}{A_v^2(0)} \int_0^\infty |A_v(\omega)|^2 d\omega \quad (45)$$

* W. W. Hansen, "Transient Response of Wide-Band Amplifiers," Proc. Nat. Electronic Conf., 1, 544-553, (1944).

which expresses the bandwidth in terms of the mean square of the absolute amplification factor. If the integral of (45) can be evaluated, the result is generally not much different from the value obtained by the previous method. Actually, the infinite integral with (43) for A_v does not exist. If we arbitrarily disregard in the numerator $j\omega C_{pg}$ we have for (45)

$$\omega_c = \frac{G^2}{g_m^2} \int_0^\infty \frac{g_m^2}{G^2 + \omega^2 C^2} d\omega = \frac{\pi G}{2 C}$$

which compares with $\sqrt{3} G/C$ just given.

As already indicated in connection with (27), the cascade connection of several identical amplifier stages simply raises the over-all single

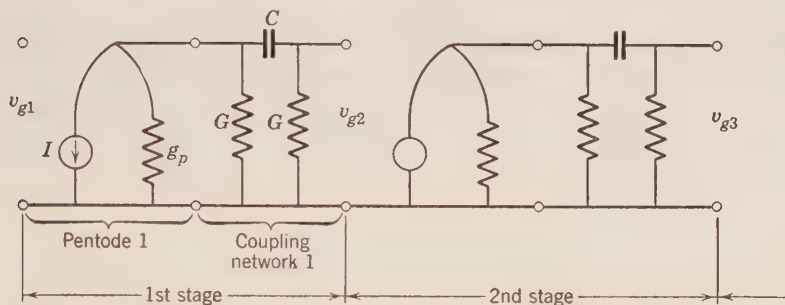


Fig. 5.10. Chain of pentode amplifiers with capacitive coupling networks.

stage matrix (27), or in more general cases (37), to the power N identical with the number of stages. Let us choose first the simpler arrangement of N stages of pentode amplifiers with capacitive coupling of the type shown in Fig. 5.7c so that the chain in Fig. 5.10 results. To simplify matters, we have disregarded the tube capacitances, assuming appropriately low frequency operation. We can express the admittance of the coupling network by

$$Y_L = G + \frac{GpC}{G + pC} = G \frac{G + 2pC}{G + pC} \quad (46)$$

For the pentode we take the matrix from (16) and (17) with $C_{pk} = C_{gk} = 0$

$$-\frac{1}{g_m} \begin{bmatrix} g_p & 1 \\ 0 & 0 \end{bmatrix}$$

so that for the complete amplifier stage we have in accordance with (27)

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix} = -\frac{1}{g_m} \begin{bmatrix} g_p + Y_L & 1 \\ 0 & 0 \end{bmatrix} \quad (47)$$

Since the pentode provides complete current isolation between stages, the voltage transfer ratio per stage is simply

$$\frac{V_{q+1}(p)}{V_q(p)} = \frac{1}{\mathfrak{A}} = -\frac{g_m}{g_p + Y_L} \quad (48)$$

where $V_q(p)$ is the Laplace transform of the input voltage and $V_{q+1}(p)$ that of the output voltage of the stage numbered q . For N stages we thus have

$$\frac{V_{N+1}(p)}{V_1(p)} = \mathfrak{A}^{-N} = (-1)^N \left(\frac{g_m}{g_p + Y_L} \right)^{-N} \quad (49)$$

This demonstrates the phase reversal, giving a negative sign if N is an odd number, and still permits any general coupling admittance Y_L .

If we now introduce the specific value of the coupling admittance (46) into (48), we can write for the voltage amplification of a single stage in parametric form

$$A_v(p) = -\frac{g_m}{g_p + G} \frac{1 + pT}{1 + \gamma pT} = \frac{1}{\mathfrak{A}} \quad (50)$$

where

$$T = \frac{C}{G}, \quad \gamma = \frac{g_p + 2G}{g_p + G} = 1 + \frac{G}{g_p + G} > 1 \quad (50a)$$

which shows at once for $p \rightarrow 0$

$$A_v(0) = -\frac{g_m}{g_p + G} \quad (51)$$

as in (30) for a simple resistive load. Because $\gamma > 1$, we expect that the bandwidth must decrease as the number of stages increases. For numerical calculations we could let $p = j\omega$ in (50) and obtain the complex voltage amplification for any number of stages as

$$A_v = [A_v(0)]^N \left(\frac{1 + j\omega T}{1 + j\omega \gamma T} \right)^N$$

To find the transient response we need to specify in (49) the applied voltage $v_1(t) = v_{g1}$ as a time function. Let us assume a simple step voltage, $v_{g1} = V_m t$; then we need to evaluate with (50) and (51)

$$v_{N+1}(t) = \mathcal{L}^{-1}[A_v(0)]^N V_m \frac{1}{p} \left(\frac{1 + pT}{1 + \gamma pT} \right)^N \quad (52)$$

Rather than try the general evaluation, which is feasible but involved, we shall obtain the inverse Laplace transform for the specific numbers $N = 1, 2$, and 3 by means of Table 1.3. For this purpose we shall rewrite the basic form of the last factor

$$\frac{1 + \gamma pT - (\gamma - 1)pT}{1 + \gamma pT} = 1 - a \frac{p}{p + \delta} \quad (53)$$

with

$$a = \frac{\gamma - 1}{\gamma}, \quad \delta = \gamma T \quad (53a)$$

Restricting ourselves to the last part in (52), since $V_m[A_v(0)]^N$ is a constant that we can consider as a scale factor, we find for

$$\begin{aligned} N = 1: \quad \mathcal{L}^{-1} \left(\frac{1}{p} - a \frac{1}{p + \delta} \right) &= 1 - ae^{-\delta t} \\ N = 2: \quad \mathcal{L}^{-1} \left(\frac{1}{p} - 2a \frac{1}{p + \delta} + a^2 \frac{p}{(p + \delta)^2} \right) &= 1 - 2ae^{-\delta t} \\ &\quad + a^2(1 - \delta t)e^{-\delta t} \\ N = 3: \quad \mathcal{L}^{-1} \left(\frac{1}{p} - 3a \frac{1}{p + \delta} + 3a^2 \frac{p}{(p + \delta)^2} - a^3 \frac{p^2}{(p + \delta)^3} \right) \\ &= 1 - (3a - 3a^2 + a^3)e^{-\delta t} - a^2(3 - 2a)\delta t e^{-\delta t} - a^3 \frac{(\delta t)^2}{2} e^{-\delta t} \end{aligned}$$

We observe the increasing number of terms as N increases, and the deterioration of the actual response as illustrated in Fig. 5.11. The increase in build-up time is, of course, a natural consequence of the decrease in bandwidth referred to previously. For the numerical calculations of the curves in Fig. 5.11 we had assumed $\gamma = 2$ which corresponds to $g_p \ll G$ in (50a).

Obviously, we can evaluate the transient solution of any chain of amplifier stages in similar fashion finding our limitation only in the complexity of the matrix of interstage networks Y_L in (47). For higher frequencies we also need to take into account the tube capacitances which will fill in the other elements of matrix (47) and make the final results more difficult to obtain. In such cases definite terminations need to be specified as well in order to make the response functions meaningful.

Rather than give other examples of this procedure, we might generalize the treatment of the amplifier chain as an *active fourpole line* in

the same manner as we had formulated the theory of the passive fourpole line in section 3.1. If we select the q 'th stage as indicated in Fig. 5.12 comprising the tube and the passive coupling network per

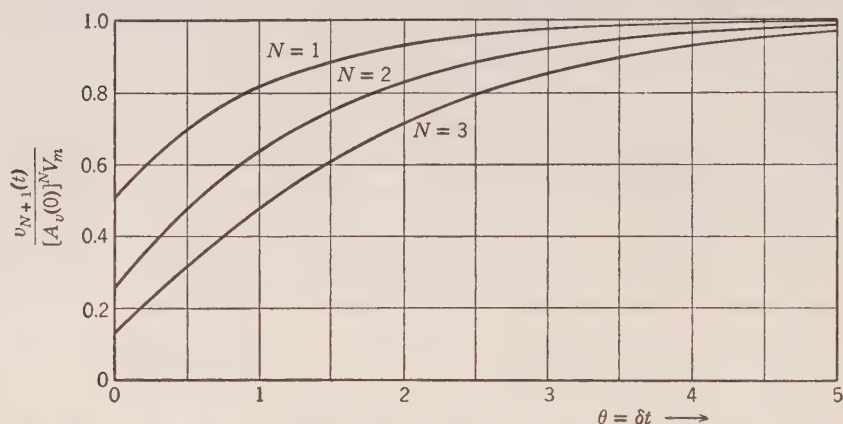


Fig. 5.11. Response of amplifier chain of Fig. 5.10 to unit step voltage for $N = 1, 2, 3$ complete stages with $\gamma = 2$.

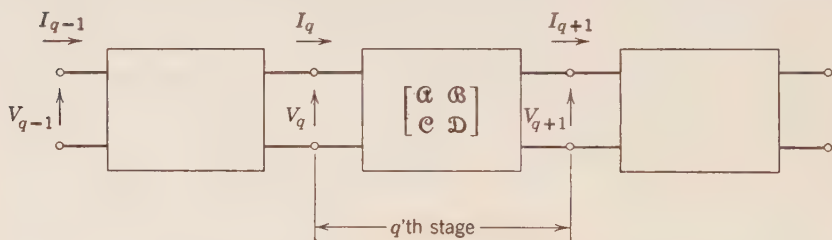


Fig. 5.12. The active fourpole line with $\alpha\delta - \beta\gamma = \eta$.

stage as in Fig. 5.10, we have the identical general relations as in (3.1)

$$\begin{aligned} V_q(p) &= \alpha V_{q+1}(p) + \beta I_{q+1}(p) \\ I_q(p) &= \gamma V_{q+1}(p) + \delta I_{q+1}(p) \end{aligned} \quad (54)$$

with the different relation

$$\alpha\delta - \beta\gamma = \eta \quad (55)$$

We can take these standard parameters $\alpha, \beta, \gamma, \delta$ from (37) if we replace indices i and o by q and $(q+1)$, respectively. All the $V(p)$ and $I(p)$ are, of course, Laplace transforms of the respective time functions. Proceeding as in section 3.1, we can combine with (54) the current expression

$$I_{q-1}(p) = \gamma V_q(p) + \delta I_q(p)$$

and eliminate in the first line of (54) $V_q(p)$ and $V_{q+1}(p)$. This gives

$$I_{q-1}(p) - (\alpha + \mathfrak{D})I_q(p) + \eta I_{q+1}(p) = 0 \quad (56)$$

Similarly we can get the voltage difference equation

$$V_{q-1}(p) - (\alpha + \mathfrak{D})V_q(p) + \eta V_{q+1}(p) = 0 \quad (57)$$

In both cases we used the active parameter relation (55) which, of course, also holds for nonbilateral fourpole lines in general, whether active or passive. We can use (56) and (57) for a gyrator line if we put $\eta = -1$ as discussed in section 2.2. The only, and apparently minor, difference between the active and passive fourpole lines is the coefficient to $I_{q+1}(p)$ in (56). Thus, the mode of solutions will be as shown in section 3.1.

We might therefore assume the general solution of (56) in the form $e^{q\Gamma}$ with Γ a nondimensional quantity, determined by the characteristic equation

$$e^{-\Gamma} - (\alpha + \mathfrak{D}) + \eta e^{\Gamma} = 0 \quad (58)$$

which we obtain directly from (56), replacing $I_q(p)$ by $e^{q\Gamma}$ and correspondingly replacing the other current transforms. Solving (58) for e^{Γ} as a quadratic equation, we obtain

$$(e^{\Gamma})_{1,2} = \frac{1}{2\eta} [(\alpha + \mathfrak{D}) \pm \sqrt{(\alpha + \mathfrak{D})^2 - 4\eta}] \quad (59)$$

or also

$$\Gamma_{1,2} = \ln \frac{(\alpha + \mathfrak{D}) \pm [(\alpha + \mathfrak{D})^2 - 4\eta]^{1/2}}{2\eta} \quad (60)$$

with the upper sign referring to index 1. We can also establish the relations

$$\eta e^{\Gamma_2} = e^{-\Gamma_1}, \quad e^{\Gamma_1 + \Gamma_2} = \frac{1}{\eta}$$

which are significantly different from (3.9) but reduce to it for $\eta = 1$. The corresponding general solution for the current transform becomes now, if we use $\Gamma \equiv \Gamma_1$ as in section 3.1

$$I_q(p) = M(\eta e^{\Gamma})^{-q} + W(e^{\Gamma})^{+q} \quad (61)$$

and for the voltage transform in analogy to (3.12)

$$V_q(p) = M'(\eta e^{\Gamma})^{-q} - W'(e^{\Gamma})^{+q} \quad (62)$$

where now the *active* iterative impedances are defined by

$$\begin{aligned} Z_{c1} &= \frac{M'}{M} = \frac{\eta e^{\Gamma} - \mathfrak{D}}{\mathfrak{C}} = \frac{1}{\mathfrak{C}} \left[\frac{\mathfrak{A} - \mathfrak{D}}{2} + \sqrt{\left(\frac{\mathfrak{A} + \mathfrak{D}}{2} \right)^2 - \eta} \right] \\ Z_{c2} &= \frac{W'}{W} = \frac{\mathfrak{D} - e^{-\Gamma}}{\mathfrak{C}} = \frac{1}{\mathfrak{C}} \left[-\frac{\mathfrak{A} - \mathfrak{D}}{2} + \sqrt{\left(\frac{\mathfrak{A} + \mathfrak{D}}{2} \right)^2 - \eta} \right] \end{aligned} \quad (63)$$

We can therefore study the active fourpole line in exactly analogous manner to the passive fourpole line. Let us apply this to the pentode amplifier chain of Fig. 5.10 for which (47) gives the parameters so that we have $\eta = 0$. (47) gives further for the iterative impedances

$$\begin{aligned} Z_{c1} &= \frac{\mathfrak{A}}{\mathfrak{C}} \rightarrow \infty \\ Z_{c2} &= \frac{\mathfrak{D}}{\mathfrak{C}} = \frac{0}{0} \rightarrow \frac{1}{g_p + Y_L} \end{aligned} \quad (64)$$

The result for Z_{c2} is obtained if we go back to the complete matrix of the pentode in (17) and use it in combination with (27). As is obvious, we have no current transmission in the forward direction which explains $Z_{c1} = \infty$. For the infinite chain we use in (62) only the first term in analogy to the passive fourpole line and observe from (59)

$$(\eta e^{\Gamma})_{\eta=0} = \mathfrak{A} + \mathfrak{D} \rightarrow \mathfrak{A} \quad (65)$$

This leads at once to the identical result established in (49) if we take the ratio of the respective voltage transforms.

For the complete chain of N like active fourpoles we can construct the over-all matrix as indicated in (3.2) and (3.3), namely

$$\begin{aligned} V_1(p) &= \mathfrak{A}_L V_{N+1}(p) + \mathfrak{B}_L I_{N+1}(p) \\ I_1(p) &= \mathfrak{C}_L V_{N+1}(p) + \mathfrak{D}_L I_{N+1}(p) \end{aligned} \quad (66)$$

where we used the indices in conformity with Fig. 5.12. The over-all matrix is defined in terms of the individual stage matrix by

$$\begin{vmatrix} \mathfrak{A}_L & \mathfrak{B}_L \\ \mathfrak{C}_L & \mathfrak{D}_L \end{vmatrix} = \begin{vmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{vmatrix}^N$$

We can evaluate the line matrix elements by introducing in (66) the corresponding expressions (61) and (62) and compare independently all terms multiplied by M and by W since these integration constants

are, indeed, independent. The final result* can be written in the form

$$\begin{vmatrix} \mathfrak{G}_L & \mathfrak{B}_L \\ \mathfrak{C}_L & \mathfrak{D}_L \end{vmatrix} = \frac{1}{Z_{C1} + Z_{C2}} \left[\frac{Z_{C1}(\eta e^\Gamma)^{N-1} + Z_{C2}e^{-(N-1)\Gamma}}{(\eta e^\Gamma)^{N-1} - e^{-(N-1)\Gamma}} - \frac{2\eta}{\mathfrak{C}} [(\eta e^\Gamma)^{N-1} - e^{-(N-1)\Gamma}] \right] \frac{Z_{C2}(\eta e^\Gamma)^{N-1} + Z_{C1}e^{-(N-1)\Gamma}}{(\eta e^\Gamma)^{N-1} - e^{-(N-1)\Gamma}} \quad (67)$$

which permits verification of all the previous results as well as reduction to the passive fourpole line parameters for $\eta = 1$.

5.4 Transistors as Active Fourpoles

For completeness, we must briefly treat transistors as recent additions to the list of active circuit elements. Fig. 5.13a indicates the more conventional p-n-p junction type or the point-contact type transistor. E is the so-called emitter electrode furnishing the positive "hole" current i_e , B is the base of normally low resistance and conventionally grounded, C the collector electrode with negative potential applied so as to draw the (amplified) collector current i_c in the direction indicated.

We have actually disregarded in the figure the biasing d-c emitter voltage maintaining the polarity of the emitter as shown, as well as the biasing d-c collector voltage. A signal voltage v_i applied at the emitter terminal pair will modulate the "hole" current i_e and directly affect the collector current i_c flowing into a load.

As a semiconductor solid, the transistor exhibits primarily a resistive circuit pattern so that the equivalent circuit as a fourpole is the simple one shown in Fig. 5.13b with the internal current source delivering a current ai_e , proportional to the emitter current. The proportionality factor a could be computed from the actual diffusion processes of holes and electrons or obtained directly by measurement of the current amplification factor as outlined later. If we desire, we can represent the active energy source also as an internal voltage source with source impedance (resistance) r_m as in Fig. 5.13c. This series voltage v_s , supporting the current i_c , is controlled by the emitter current, so that its value is given by

$$v_s = r_m i_e = r_c (ai_e) \quad (68)$$

where the last term refers to the current source in Fig. 5.13b which

* Similar results, though apparently with η missing in some places, are given in Brown and Bennett, *loc. cit.*

causes the equivalent voltage drop across the collector resistance r_c . From (68) we take as definition

$$a = \frac{r_m}{r_c} \quad (68a)$$

We need to observe here that in the physical arrangement the collector signal voltage v_c is directed like a "voltage drop" in any passive load and thus agrees with the convention of our fourpole

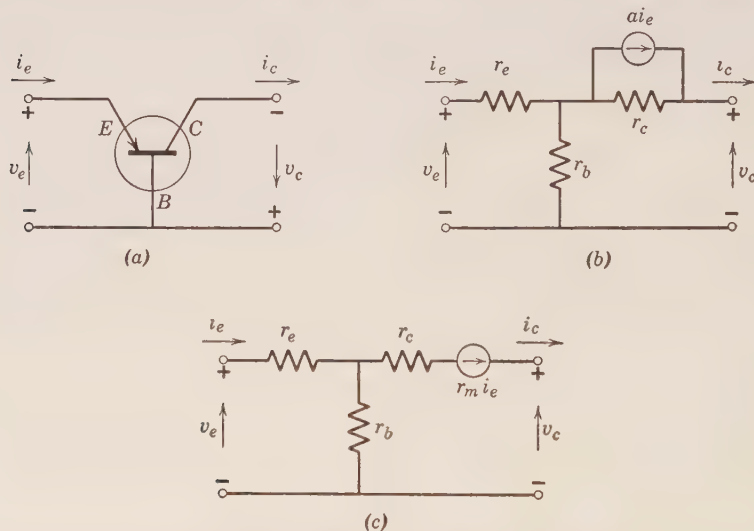


Fig. 5.13. The grounded-base transistor (a) and its equivalent circuit (b) with internal current source, (c) with internal voltage source.

notation in Figs. 5.13b and c where the direction indicates that it acts like a source voltage upon any network connected in cascade. We will therefore not expect the phase reversal that we found in electron tubes, section 5.1. Actually, it will be advantageous to think of the transistor primarily as a current amplification device through direct conductive action, whereas the electron tube could be considered as primarily a voltage amplification device. The latter becomes most impressively demonstrated in the pentode amplifier chain, Fig. 5.10.

Since the equivalent circuit of the transistor appears as a resistive T, we might most readily apply the mesh analysis. Writing the Kirchhoff equations for the equivalent circuit in Fig. 5.13c with i_e and i_c as mesh currents

$$\begin{aligned} v_e &= i_e(r_e + r_b) - i_c r_b \\ i_e r_m - v_c &= i_c(r_b + r_c) - i_e r_b \end{aligned} \quad (69)$$

We use $i_e r_m$ as the driving voltage in the direction of the mesh current (voltage source) and v_c as the generated output terminal voltage.* As fourpole we want to express the input pair v_e, i_e in terms of the output pair v_c, i_c . The second line gives directly

$$i_e = \frac{v_c}{r_b + r_m} + \frac{r_b + r_c}{r_b + r_m} i_c \quad (70)$$

and using this in the first line, we obtain

$$v_e = \frac{r_b + r_e}{r_b + r_m} v_c + \left(\frac{(r_b + r_c)(r_b + r_e)}{r_b + r_m} - r_b \right) i_c \quad (71)$$

We can replace by (68a) r_m by ar_c and thus write the standard parameter matrix for the transistor with grounded base

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{\text{Tr,gb}} = \frac{1}{r_b + ar_c} \begin{bmatrix} r_b + r_e & r_e(r_b + r_c) + (1 - a)r_b r_c \\ 1 & r_b + r_c \end{bmatrix} \quad (72)$$

which is purely real. This matrix is again asymmetrical and its determinant has the simple value

$$\mathfrak{A}\mathfrak{D} - \mathfrak{B}\mathfrak{C} = \frac{r_b}{r_b + ar_c} = \eta \quad (73)$$

which will generally be very small since the base resistance is very small compared with ar_c . In a formal way, (73) compares very closely with (13) for the conventional triode which we shall note again later on.

From (70) we find the short-circuit current amplification factor

$$(A_i)_{sc} = \left(\frac{i_c}{i_e} \right)_{v_c=0} = \frac{r_b + r_m}{r_b + r_c} = \frac{a + (r_b/r_c)}{1 + (r_b/r_c)} \approx a \quad (74)$$

Since generally the base resistance $r_b \ll r_c$, we can approximate this amplification factor by the current source parameter a . This permits ready experimental determination of this important parameter as indicated in the foregoing.

Earlier attempts to draw analogies between transistors and electron tubes have not been particularly successful because of the difference in basic physical characteristics. As pointed out, the transistor behaves as a current amplifying device. Let us now construct the dual to the equivalent circuit in Fig. 5.13c by the method explained in detail in Vol. I, section 3. We select in the center of the two meshes

* M. J. E. Golay, "The Equivalent Circuit of the Transistor," *Proc. I.R.E.*, **40**, 360 (1952).

in Fig. 5.14a points A and B which become nodes in the dual network, Fig. 5.14b, and we add the external point D which becomes the common reference node in the dual. With the normalization factor r^2 , we have then the following relations between corresponding quantities in the dual networks:

$$\begin{aligned} r_e &= r^2 g_e & r i_1' &= v_1 \\ r_c &= r^2 g_c & r i_2' &= v_2 \\ r_b &= r^2 g_b & r i_s &= v_s \end{aligned}$$

This permits us to carry through numerical calculations in either of the dual networks and to transfer these to the other. Now, if we look

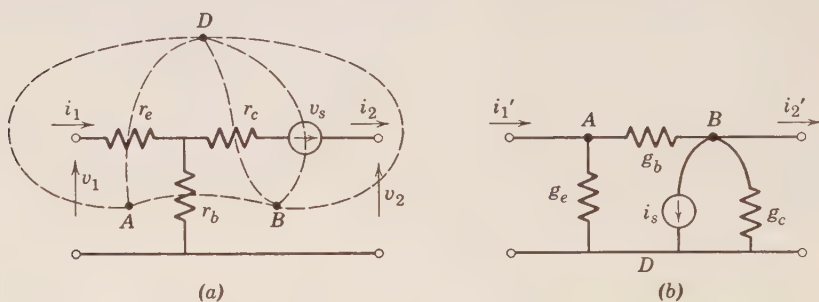


Fig. 5.14. Construction of the dual network (b) to the equivalent circuit (a) of the transistor.

at Fig. 5.14b and compare with the equivalent circuit of the conventional triode, Fig. 5.1d, we find a close analogy. For the junction transistor we can assume that both r_e and r_b are small compared with r_c , so that also g_e and g_b will be small compared with g_c . This makes the analogy so close that we can consider* the *triode* at low frequencies, i.e., with neglect of the capacitances, the *approximate dual of the junction transistor*!

Indeed, if we let $g_e = g_b = 0$ in Fig. 5.14b, it becomes identical with either Fig. 5.1d or with the pentode equivalent circuit which we used in Fig. 5.10. This suggests looking at the fourpole parameters. Thus, if we let for the junction transistor $r_b = r_e = 0$, then we have instead of (72) the simpler matrix

$$\frac{1}{ar_c} \begin{bmatrix} 0 & 0 \\ 1 & r_c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1/ar_c & 1/a \end{bmatrix}$$

* R. L. Wallace, Jr., and G. Raisbeck, "Duality as a Guide in Transistor Design," *Bell System Tech. J.*, **30**, 381-417 (1951).

in contrast to the singular pentode matrix from (17), letting there $C_{pk} \rightarrow 0$

$$\frac{1}{-\mu g_p} \begin{bmatrix} g_p & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1/\mu & -1/\mu g_p \\ 0 & 0 \end{bmatrix}$$

which we used in (47). The similarity clearly suggests a dual relationship between g_p of the pentode and r_c of the transistor, as well as of $(-\mu)$, the voltage amplification factor of the pentode (basically open circuit) and the short-circuit current amplification factor of a transistor. We also see the phase reversal of the single-stage tube amplifier in the negative sign of μ . Finally, we observe that the simplified transistor matrix has reduced to the elements defining input current, whereas the simplified pentode matrix (or triode matrix) has reduced to the elements defining the input voltage, bearing out the suggested basic characterization of these two types of active elements.

The fact that a *dual* relationship exists between tubes and transistors can be used as a guide in circuit design for amplifier, oscillator, modulator, and detector operation. Rather than combine the transistor with the identical circuits used for electron tubes, we must combine them with the duals of these.*

As with the triode, the transistor can be used in different circuit arrangements such as grounded-emitter and grounded-collector configurations. Rather extensive treatments† have been given of these as well as of multistage amplifier circuits with different coupling networks. These combinations can be readily handled for steady-state calculations in the same manner as the tube circuits and with the same ease if we use the matrix notation given in the foregoing and its appropriate modifications.

For higher frequencies, the equivalent circuit of the transistor must be modified to account for the capacitive effect of the barrier layer near the collector. A simple modification is the introduction of a capacitance C_c across the collector equivalent resistance r_c and the voltage source as shown in Fig. 5.15. If we designate the parallel combination of collector resistance and capacitance as collector impedance $Z_c(p)$ in parametric form

$$Z_c(p) = \frac{r_c}{1 + r_c C_c p} \quad (75)$$

* Wallace and Raisbeck, *loc. cit.* Also G. Raisbeck, "Transistor Circuit Design," *Electronics*, **24**, 128-132, 134 (1951).

† R. F. Shea, *Principles of Transistor Circuits*, Wiley, New York, 1953.

then the voltage source will deliver in Laplace transform notation

$$V_s(p) = a(p)Z_c(p)I_e(p) \quad (76)$$

in analogy to the low frequency value (68). The factor $a(p)$ had been identified as the short-circuit current amplification factor in (74).

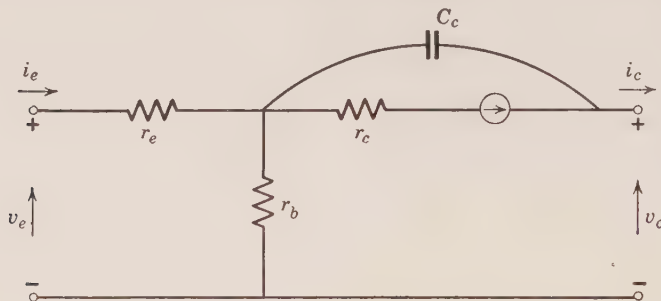


Fig. 5.15. The equivalent circuit of the grounded-base transistor for higher frequencies.

For higher frequencies, $a(p)$ must portray the time constant of the diffusion process and a delay which can be combined by defining*

$$a(p) = \frac{a_0}{1 + T_c p} e^{-\tau p} \quad (77)$$

The delay time τ as well as the time constant T_c could be computed from a solution of the diffusion process, but they are generally determined from measurements.

The Kirchhoff equations (69) must now be written in terms of the Laplace transforms, namely

$$\begin{aligned} V_e(p) &= (r_e + r_b)I_e(p) - r_b I_c(p) \\ V_s(p) - V_c(p) &= (r_b + Z_c)I_c(p) - r_b I_e(p) \end{aligned} \quad (78)$$

The fourpole parameters for the grounded-base transistor become now, proceeding as in (70) and (71)

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{\text{Tr,gb}} = \frac{1}{r_b + aZ_c} \begin{bmatrix} r_b + r_e & r_e(r_b + Z_c) + (1 - a)r_b Z_c \\ 1 & r_b + Z_c \end{bmatrix} \quad (79)$$

These parameters are therefore a rather obvious and simple extension from the low frequency case. We also can verify that the equivalent of (74) holds if we take the second line of (78) and introduce $V_s(p)$ from (76), namely

* *Principles of Transistor Circuits, op. cit., p. 382.*

$$A_i(p)_{V_c=0} = \left(\frac{I_e(p)}{I_c(p)} \right)_{V_c=0} = \frac{aZ_c + r_b}{Z_c + r_b} \rightarrow a(p)$$

To select a simple demonstration for the transient response, let us take the idealized transistor as the dual of the pentode and cascade it with a lossless parallel tuned circuit as coupling network as shown in Fig. 5.16. The complete parameter matrix for the amplifier stage is

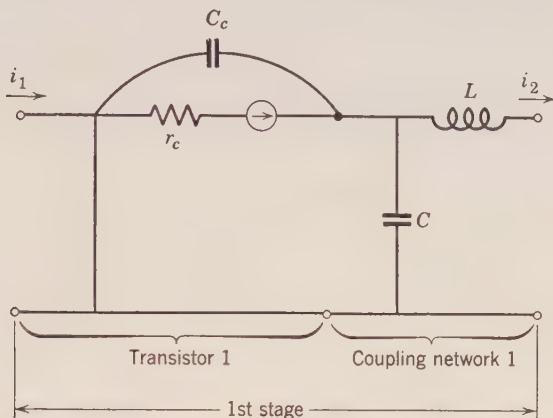


Fig. 5.16. Transistor amplifier as dual of pentode with parallel tuned circuit as coupling network.

given in terms of Laplace transforms by the product of the transistor matrix (79) for $r_e = r_b = 0$ and the matrix of the passive fourpole

$$\begin{bmatrix} V_1(p) \\ I_1(p) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ (aZ_c)^{-1} & a^{-1} \end{bmatrix} \times \begin{bmatrix} 1 & pL \\ pC & 1 + p^2LC \end{bmatrix} = \begin{bmatrix} V_2(p) \\ I_2(p) \end{bmatrix} \quad (80)$$

For the current transform ratio, since $V_2(p) = 0$ because of the short-circuit input to stage 2, we need only the lower right-hand term of the matrix product and that is

$$\frac{I_1(p)}{I_2(p)} = \frac{1}{a(p)} \left(\frac{pL}{Z_c} + 1 + p^2LC \right)$$

If we introduce (75) and (77), and specify the input current as the step function $I_m t$ with the Laplace transform I_m/p , we obtain for the output current transform

$$I_2(p) = \frac{a_0 e^{-\tau p}}{1 + T_c p} \frac{1}{p^2(LC^*) + p \frac{L}{r_c} + 1} \frac{I_m}{p} \quad (81)$$

where $C^* = C_c + C$. The inversion into time functions is rather straightforward since we can readily apply the residue theorem or expansion theorem. The exponential factor $e^{-\tau p}$ indicates a time delay which we can take care of by line 2 of Table 1.4. We thus have as the complete solution for the current step response

$$i_2(t) = a_0 I_m \left(1 - \frac{(\Omega_0 T_c)^2 e^{-\frac{t'}{T_c}}}{1 - 2\delta T_c + (\Omega_0 T_c)^2} - \frac{\Omega_0^2 e^{-\delta t'} \sin(\Omega t' + \varphi' - \varphi'')}{2\Omega \sqrt{\delta^2 + \Omega^2} \sqrt{(1 - \delta T_c)^2 + (\Omega T_c)^2}} \right) \quad t' > 0 \quad (82)$$

where we used the abbreviations

$$\begin{aligned} 2\delta &= \frac{1}{r_c C^*} & \tan \varphi' &= \frac{\Omega}{\delta} \\ \Omega_0^2 &= \frac{1}{LC^*} & \tan \varphi'' &= \frac{\Omega T_c}{1 - \delta T_c} \\ \Omega &= \Omega_0 \sqrt{1 - \left(\frac{\delta}{\Omega_0}\right)^2} & t' &= t - \tau \end{aligned}$$

The first term gives the amplified current step delayed by time τ as the ideal low frequency response. The second term portrays the effect of the high frequency collector capacity and diffusion time constant interacting with the coupling network parameters. The last term is the direct effect of the parallel resonant circuit. Because we disregarded the normal coil losses, we must expect oscillations damped only by the collector resistance. With appropriate damping, however, e.g., adding resistance R in series with L , this term can assist in counteracting the second term and in improving the transient response.

Extension to multistage amplifiers of various configurations is self-evident. Similarly, we could use the relations deduced for the general active fourpole line in (61) and (62), as well as in (67).

5.5 Feedback Amplifiers

So far we have considered only cascade connections of networks in detail. If active and passive networks are connected in series or in parallel, or in any combination of series-parallel, the output and input signals become interrelated by "feedback." We shall consider here only the principles of feedback systems as far as they present special

illustrations of linear transient analysis and we shall not delve into the detailed characteristics of feedback systems, their analysis and design. Section B of Appendix 3 gives a list of treatises and textbooks which may be consulted for such details.

Let us first consider the simple schematic arrangement of Fig. 5.17 where the so-called forward active network or amplifier N has the over-all input and output voltage-current pair Laplace transforms $V_i(p)$, $I_i(p)$ and $V_o(p)$, $I_o(p)$. The feedback network F , generally assumed to be a passive network, is presumably connected with its input terminal pair to the output terminals of N , whereas its output terminal pair is interconnected at the input terminal pair of N , thus

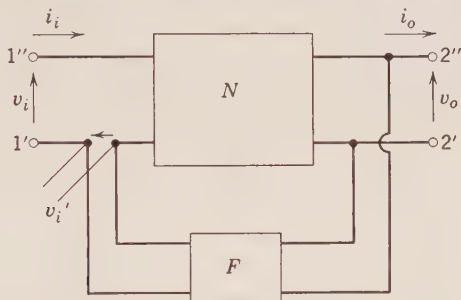


Fig. 5.17. Schematic series-parallel type of feedback system; N forward active network or amplifier, F feedback network.

completing a “feedback loop.” In Fig. 5.17 network F is connected in parallel at $2'2''$, whereas it is connected in series at the input of N , constituting a series-parallel interconnection of the two networks.

According to the elementary feedback theory, e.g., Bode,^{B2} p. 32, we can argue in this example in the following manner: without the feedback loop, the original amplifier N will exhibit a voltage amplification ratio $A_v^0(p)$ which in general is computed as the ratio

$$A_v^0(p) = \left(\frac{V_o(p)}{V_i(p)} \right)_0 \rightarrow \mu \quad (83)$$

the index 0 indicating no feedback. Obviously, as pointed out in section 5.2, particularly (29), if we put $p = j\omega$, we obtain the complex amplification ratio and at very low frequencies it reduces for the triode to the real d-c amplification factor ($-\mu$) so that Bode^{B2} has used this designation μ for any condition of operation. The passive network F might be characterized by a general transfer function $\beta(p)$, so that the output voltage of F can be written

$$\beta(p)V_o(p) = V_i'(p) \quad (84)$$

Again, for steady-state a-c analysis $\beta(j\omega)$ will be the conventional a-c transfer function of F . If we assume for this simplified theory that the interconnection of the two networks in series parallel does not affect their isolated characteristics A_v^0 and β , then we can state that the combined input voltage to N must remain related to the output voltage as in (83), and we can write

$$[V_i(p) + V_i'(p)]A_v^0(p) = V_o(p)$$

Because of (84), we can rearrange the terms and obtain

$$V_o(p) = \frac{A_v^0}{1 - \beta A_v^0} V_i(p) \rightarrow \frac{-\mu}{1 + \beta\mu} V_i \quad (85)$$

the classical feedback relation.* Though this simple relation portrays rather graphically the basic relationships, the underlying simplifying assumptions are not always warranted. The rigorous solution of the circuit problem will generally give a rather different form, but we can, with some imagination, often reduce it to the simplicity of (85).

As a simple concrete example, take the voltage feedback circuit of Fig. 5.18a. Applying the simple feedback theory, we determine the amplification ratio for the triode with impedance load Z_l , disregarding entirely the feedback connections as in Fig. 5.18c. From (29) we can deduce, suppressing the capacitances

$$A_v^0(p) = \frac{-\mu g_p}{g_p + Y_l(p)} = \frac{-\mu Z_l}{Z_l + r_p}$$

The feedback circuit takes the fraction $\beta = R_1/(R_1 + R_2)$ of the output voltage and applies it in series at the input terminals. Thus, we can write down at once for the voltage ratio with feedback from (85)

$$\frac{V_o(p)}{V_i(p)} = \frac{-\mu Z_l}{(Z_l + r_p) - \beta(-\mu Z_l)} = \frac{-\mu Z_l}{r_p + Z_l + Z_l\beta\mu} \quad (86)$$

indicating in general a lower amplification ratio, as one would expect with negative feedback which generally tends to stabilize the amplifier.

Let us now carry through the rigorous analysis by means of the equivalent fourpole interconnection indicated in Fig. 5.18b. The simplified fourpole parameters of the triode with terminal pairs 3'3'' and 4'4'' are given by (5), to wit

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_N = -\frac{1}{\mu} \begin{bmatrix} 1 & r_p \\ 0 & 0 \end{bmatrix} \quad (87)$$

* H. S. Black, "Stabilized Feedback Amplifiers," Bell System Tech. J., **13**, 1-18 (1934); also in *Elec. Eng.*, **53**, 114-120 (1934).

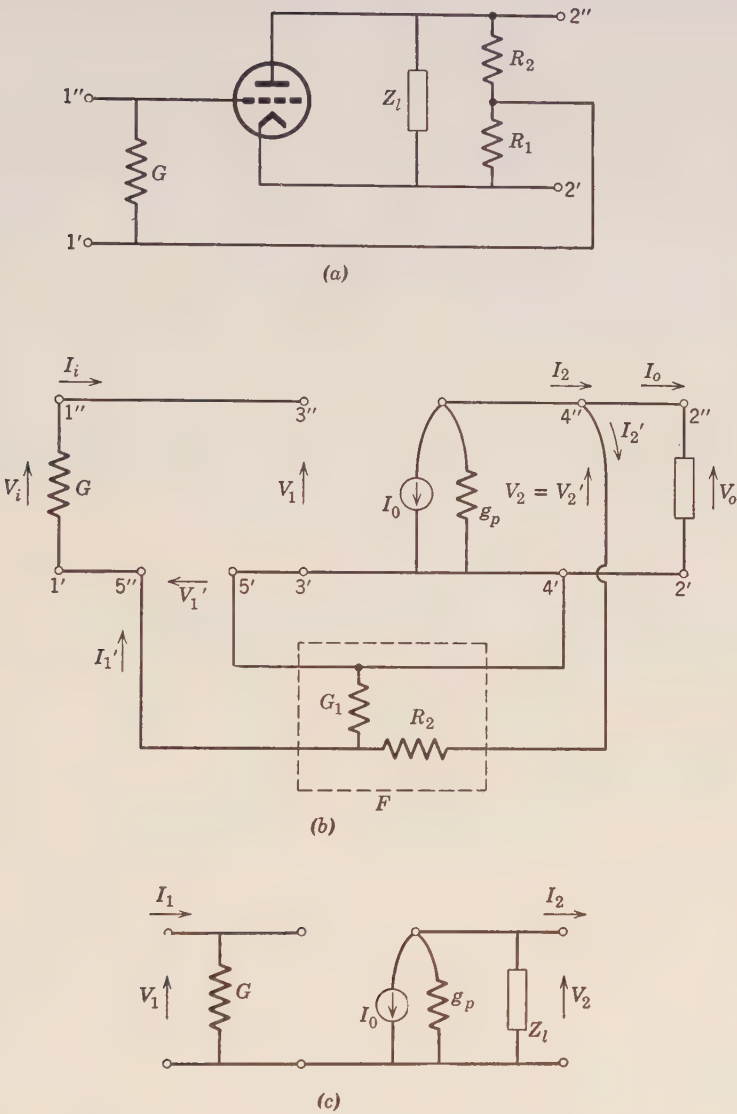


Fig. 5.18. Single triode amplifier with voltage feedback. (a) Actual circuit; (b) series-parallel connection of corresponding active and passive networks; (c) triode without feedback circuit.

The feedback circuit is the cascade connection of conductance G_1 and resistance R_2 , and the combination matrix is

$$\begin{bmatrix} 1 & 0 \\ G_1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & R_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & R_2 \\ G_1 & G_1 R_2 + 1 \end{bmatrix} = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_F \quad (88)$$

To find the over-all parameters of the series-parallel combination of two fourpoles, we need to introduce specific designations for the voltage-current pairs. Suppose we use as in Fig. 5.18b

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_N \times \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad (89)$$

for the active network, and

$$\begin{bmatrix} V_1' \\ I_1' \end{bmatrix} = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_F \times \begin{bmatrix} V_2' \\ I_2' \end{bmatrix} \quad (90)$$

for the passive feedback network. The interconnection demands the relations

$$\begin{aligned} V_i &= V_1 - V_1' & V_o &= V_2 = V_2' \\ I_i &= I_1 = I_1' & I_o &= I_2 - I_2' \end{aligned} \quad (91)$$

If we therefore take the first lines of (89) and (90) in expanded form and subtract them, we obtain with recognition of (91)

$$\begin{aligned} V_i &= V_1 - V_1' = \mathfrak{A}_N V_2 + \mathfrak{B}_N I_2 - \mathfrak{A}_F V_2' - \mathfrak{B}_F I_2' \\ &= (\mathfrak{A}_N - \mathfrak{A}_F) V_o + \mathfrak{B}_N I_2 - \mathfrak{B}_F I_2' \end{aligned}$$

We then can write the second lines of (89) and (90)

$$\begin{aligned} I_i &= I_1 = \mathfrak{C}_N V_2 + \mathfrak{D}_N I_2 = \mathfrak{C}_N V_o + \mathfrak{D}_N I_2 \\ I_i &= I_1' = \mathfrak{C}_F V_2' + \mathfrak{D}_F I_2' = \mathfrak{C}_F V_o + \mathfrak{D}_F I_2' \end{aligned}$$

This permits direct evaluation of I_2 and I_2' in terms of I_i and V_o so that

$$I_o = I_2 - I_2' = \frac{\mathfrak{D}_F - \mathfrak{D}_N}{\mathfrak{D}_F \mathfrak{D}_N} I_i - \frac{\mathfrak{C}_N \mathfrak{D}_F - \mathfrak{C}_F \mathfrak{D}_N}{\mathfrak{D}_F \mathfrak{D}_N} V_o$$

We obtain now the final forms

$$\begin{aligned} V_i &= \left((\mathfrak{A}_N - \mathfrak{A}_F) + \frac{(\mathfrak{B}_N - \mathfrak{B}_F)(\mathfrak{C}_N - \mathfrak{C}_F)}{\mathfrak{D}_F - \mathfrak{D}_N} \right) V_o \\ &\quad + \frac{\mathfrak{B}_N \mathfrak{D}_F - \mathfrak{B}_F \mathfrak{D}_N}{\mathfrak{D}_F - \mathfrak{D}_N} I_o \quad (92) \end{aligned}$$

$$I_i = \frac{\mathfrak{C}_N \mathfrak{D}_F - \mathfrak{C}_F \mathfrak{D}_N}{\mathfrak{D}_F - \mathfrak{D}_N} V_o + \frac{\mathfrak{D}_N \mathfrak{D}_F}{\mathfrak{D}_F - \mathfrak{D}_N} I_o$$

which define the over-all parameters of the series-parallel combination

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{s-p}$$

in terms of the individual parameters of the isolated networks.

If we apply this now to the over-all feedback loop we need only use the first relation of (92) with the further condition that $V_o = I_o Z_l$, so that

$$V_i = \mathfrak{A}_{s-p} V_o + \mathfrak{B}_{s-p} \frac{V_o}{Z_l}$$

The voltage amplification ratio thus becomes for the feedback amplifier

$$A_v = \frac{V_o}{V_i} = \frac{Z_l}{Z_l \mathfrak{A}_{s-p} + \mathfrak{B}_{s-p}} \quad (93)$$

Computing the parameters with the aid of (92) and the values from (87) and (88), we get

$$\mathfrak{A}_{s-p} = \frac{G_1(R_2 - r_p) + 1 + \mu}{-\mu(G_1 R_2 + 1)}$$

$$\mathfrak{B}_{s-p} = -\frac{r_p}{\mu}$$

and therefore

$$A_v = \frac{-\mu Z_l}{r_p + Z_l + \beta Z_l [\mu - (r_p/R_1)]} \quad (94)$$

This looks quite similar to (86), except that instead of $\beta\mu$ in the last term of the denominator in (86) we have here

$$\beta\mu \left(1 - \frac{r_p}{\mu R_1} \right)$$

which is the correction to the feedback factor required by the interaction of the simultaneous presence of the two networks. In this elementary case we can readily compare the results of simple and rigorous treatments, but it is not so easy in more complicated systems. However, where accuracy is required it is necessary to carry through the more complete analysis, particularly for the transient performance.

We can now study the transient response of this feedback circuit, say, for a step voltage $V_m t$ applied to it. To specify the load impedance simply, take $Z_l = (pC)^{-1}$, a pure capacitance. This gives with (94)

$$V_o(p) = A_v \frac{V_m}{p} = -\mu \frac{V_m}{T} \frac{1}{p(p + 1/\alpha T)}$$

where we introduced

$$T = r_p C, \quad \alpha = 1 + \beta \left(\mu - \frac{r_p}{R_1} \right) \gtrless 0$$

The value of α could conceivably be chosen positive or negative depending upon the choice of R_1 . The inverse Laplace transform is

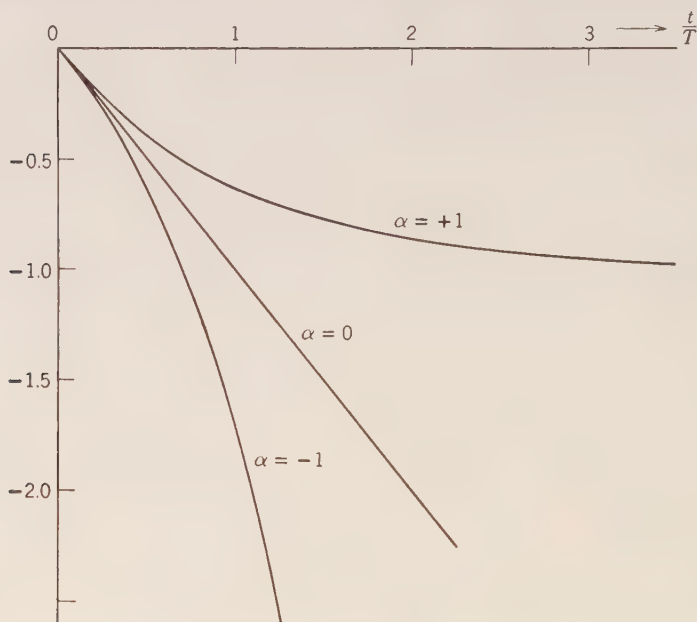


Fig. 5.19. Response of voltage feedback amplifier to step voltage for three values of $\alpha = 1 + \beta[\mu - (r_p/R_1)]$, namely $\alpha = +1, 0, -1$.

found readily in line 5, Table 1.3, and gives for $\alpha > 0$

$$v_o(t) = -\alpha \mu V_m (1 - e^{-t/\alpha T}) \quad \alpha > 0$$

the usual build-up of a voltage across a capacitance in the negative direction because of the phase reversal in the triode. For $\alpha = 0$, we need to return to (94) in order to see the effect and we find

$$V_o(p) = -\frac{\mu}{T} V_m \frac{1}{p^2}$$

so that

$$v_o(t) = -\mu V_m \frac{t}{T} \quad \alpha = 0$$

which we would certainly not consider a desirable and stable response. If then $\alpha < 0$, we can return to the first form, define $\alpha = -|\alpha|$, so as to set in evidence the negative real value of α , and find

$$v_o(t) = |\alpha| \mu V_m (1 - e^{t/|\alpha|T}) \quad \alpha < 0$$

Figure 5.19 shows the three types of response graphically. We shall discuss the problem of stability in the next section.

As another and different example, we might choose shunt feedback, as in Fig. 5.20, demonstrating the value of the systematic matrix notation and illustrating some more difficult aspects of the elementary theory of feedback. Let us first apply the appropriately modified

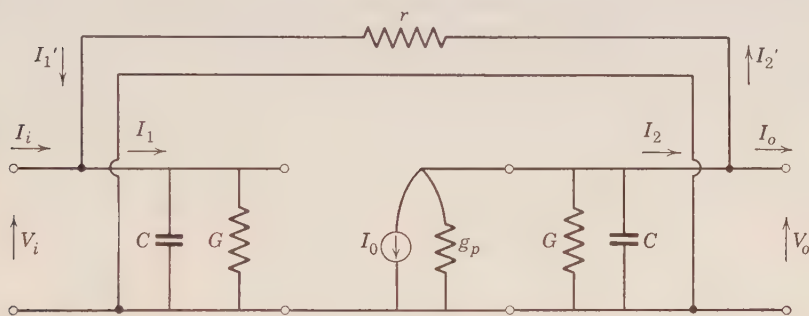


Fig. 5.20. Local shunt feedback or parallel combination of active and passive networks.

elementary feedback theory. Having essentially current feedback, we shall deal with current relations. We assume that without feedback we have determined the current amplification ratio

$$A_i^0(p) = \left(\frac{I_2(p)}{I_1(p)} \right)_0 \quad (95)$$

similar to (83) in the voltage feedback case. The fraction I_2' of the output current I_2 which is fed back we might designate in Laplace transform as

$$I_2'(p) = \gamma(p) I_2(p), \quad I_1'(p) = \beta(p) I_2(p)$$

and correspondingly the fraction that comes back into the input terminals. In our case $\gamma = \beta$, but this need not always be the case. The actual evaluation of either γ or β is generally very difficult, because it requires the complete solution of the current distribution in the networks.

The input current can be written, also in Laplace transforms, for the general case

$$I_i(p) = I_1 - I_1' = [A_i^0(p) - \beta(p)]I_2(p)$$

and we also have

$$I_2(p) = I_o - I_2' = I_o - \gamma I_2$$

Thus, the over-all current amplification ratio with feedback becomes

$$\frac{I_o(p)}{I_i(p)} = A_i(p) = \frac{A_i^0}{1 - \beta A_i^0} (1 - \gamma) \quad (96)$$

which has a structure quite similar to (85).

The rigorous solution is obtained best by constructing the over-all fourpole parameters for the parallel combination of the active network N , for which we have the cascading three fourpoles

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ G + pC & 1 \end{bmatrix} \times \left(-\frac{1}{\mu} \right) \begin{bmatrix} 1 & r_p \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ G + pC & 1 \end{bmatrix} = \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

and the passive feedback network F , in our case a simple resistance

$$\begin{bmatrix} V_1' \\ I_1' \end{bmatrix} = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} V_2' \\ I_1' \end{bmatrix}$$

The terminal conditions in this case as we read from Fig. 5.20 are

$$\begin{aligned} V_1' &= V_1 = V_i & V_2' &= V_2 = V_o \\ I_i &= I_1 - I_1' & I_o &= I_2 - I_2' \end{aligned} \quad (97)$$

Though similar, they are significantly different from (91). As there, we might use the subscripts N and F and employ general parameter notation to make the results usable in all instances. We have

$$V_i = \mathfrak{A}_N V_2 + \mathfrak{B}_N I_2 = \mathfrak{A}_F V_2' + \mathfrak{B}_F I_2' = V_1 = V_1'$$

$$I_1 = \mathfrak{C}_N V_2 + \mathfrak{D}_N I_2$$

$$I_1' = \mathfrak{C}_F V_2 + \mathfrak{D}_F I_2$$

Combining with relations (97) we can finally express

$$\begin{aligned} V_i &= \frac{\mathfrak{A}_N \mathfrak{B}_F + \mathfrak{A}_F \mathfrak{B}_N}{\mathfrak{B}_N + \mathfrak{B}_F} V_o + \frac{\mathfrak{B}_N \mathfrak{B}_F}{\mathfrak{B}_N + \mathfrak{B}_F} I_o \\ I_i &= \left((\mathfrak{C}_N + \mathfrak{C}_F) + \frac{(\mathfrak{D}_N - \mathfrak{D}_F)(\mathfrak{A}_F - \mathfrak{A}_N)}{\mathfrak{B}_N + \mathfrak{B}_F} \right) V_o \\ &\quad + \frac{\mathfrak{D}_N \mathfrak{B}_F + \mathfrak{B}_N \mathfrak{D}_F}{\mathfrak{B}_N + \mathfrak{B}_F} I_o \end{aligned} \quad (98)$$

from which we read the parameters of the parallel combination

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{\text{par}}$$

Returning to our special case, we compute the parameters of the N network by normal matrix multiplication, keeping the matrices in the order of appearance and abbreviating $G + pC = Y$

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_N = -\frac{1}{\mu g_p} \begin{bmatrix} Y + g_p & 1 \\ Y(Y + g_p) & Y \end{bmatrix}$$

Combining these now with the parameters for the F network as needed for (98), we can construct the complete matrix for the parallel network. However, to find the current amplification ratio (96) we need not go through all that. Since a current ratio is required we must specify a termination, say a resistance load R_2 across the output terminals, so that $V_o = R_2 I_o$. Then the second relation in (98) permits us at once to write

$$I_i = \mathfrak{C}_{\text{par}} V_o + \mathfrak{D}_{\text{par}} I_o = (\mathfrak{C}_{\text{par}} R_2 + \mathfrak{D}_{\text{par}}) I_o$$

and we have

$$\frac{I_o}{I_i} = \frac{1}{\mathfrak{C}_{\text{par}} R_2 + \mathfrak{D}_{\text{par}}} \quad (99)$$

For any other load Z_l we just need to replace R_2 by Z_l . Carrying through the indicated combinations for $\mathfrak{C}_{\text{par}}$ and $\mathfrak{D}_{\text{par}}$, we can simplify to the final form

$$\frac{I_o(p)}{I_i(p)} = \frac{1 - \mu g_p r}{1 + rY + R_2[2Y + g_p + rY(Y + g_p)] + \mu g_p R_2} \quad (100)$$

If on the other hand we formulate A_i^0 from the N network parameters in analogy to (99), namely

$$A_i^0(p) = \frac{1}{\mathfrak{C}_N R_2 + \mathfrak{D}_N} = \frac{-\mu g_p}{Y(1 + R_2(Y + g_p))} = \frac{-\mu g_p}{Y^*}$$

we can write (100) formally similar to (96)

$$\begin{aligned} \frac{I_o}{I_i} = A_i(p) &= \frac{A_i^0 + \frac{1}{rY^*}}{1 - A_i^0(R_2 \rho Y^*)} (\rho r Y^*) \\ \rho^{-1} &= 1 + R_2(2Y + g_p) + rY^* \end{aligned} \quad (100a)$$

In order to reduce (100a) somewhat closer to the form of (96) we would need to let $rY^* \rightarrow \infty$ in the numerator and define

$$\beta(p) = R_{2\rho}Y^*$$

which is a rather involved relation as we would expect from the greater complexity of the network.

We observe that (100) would easily permit us to solve for the transient response because the denominator is only quadratic in p . Since only positive coefficients occur in this quadratic, we conclude that the response can only lead to roots with negative real parts and thus to an absolute stable performance. We also can check (100) readily if we let the capacitances vanish so that $Y \rightarrow G$ and solve for the current and voltage values directly from the Fig. 5.20.

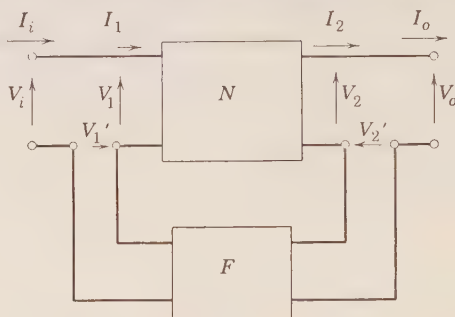


Fig. 5.21. Series combination of active and passive networks of series feedback.

For completeness we might also give the combination parameters for series feedback, or series combination of an active and a passive network as shown in Fig. 5.21

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{\text{ser}} = \frac{1}{\mathfrak{C}_N + \mathfrak{C}_F} \begin{bmatrix} \mathfrak{A}_N \mathfrak{C}_F + \mathfrak{A}_F \mathfrak{C}_N & (\mathfrak{B}_N + \mathfrak{B}_F)(\mathfrak{C}_N + \mathfrak{C}_F) + (\mathfrak{D}_N - \mathfrak{D}_F)(\mathfrak{A}_F - \mathfrak{A}_N) \\ \mathfrak{C}_N \mathfrak{C}_F & \mathfrak{D}_N \mathfrak{C}_F + \mathfrak{D}_F \mathfrak{C}_N \end{bmatrix} \quad (101)$$

The elementary feedback theory leads with the definitions

$$V_i = V_1 + V_1' = V_1 + \beta V_2$$

$$V_o = V_2 + V_2' = (1 + \gamma) V_2$$

to the voltage amplification ratio

$$\frac{V_o(p)}{V_i(p)} = \frac{A_v^0}{1 + A_v^0 \beta} (1 + \gamma) \quad (102)$$

which is quite similar to (96) for the current amplification ratio in shunt feedback and could be considered its dual.

5.6 Stability of Active Networks

In dealing with finite purely passive networks we had emphasized that they were inherently stable, i.e., that all modes of the free response of actual physical networks had positive damping coefficients, that lossless networks might sustain continuous oscillations, but that no passive network, if left to itself, would build its response to indefinite magnitude. The general proof would proceed from the nature of a network function, either impedance, admittance, or transfer function, and demonstrate that any of these can be represented as the fraction of two *Hurwitz polynomials* in p ; that a Hurwitz polynomial is one with all real coefficients and with its zeros lying in the left-half p -plane.

In active networks, because of the presence of an internal energy source, it becomes very important to establish criteria for the stability of the system in terms of the network parameters. The earliest general criterion was deduced by E. J. Routh, and reiterated on a different basis by Hurwitz. A rather detailed account is given in Guillemin;* here only a brief account is given for practical applications.

It appears obvious that as long as we deal with rational Laplace transforms in our solution of these network problems, the stability of the solution can be judged by the presence or absence of poles of the Laplace transform in the right-half p -plane. The difficulty is encountered in establishing this fact without carrying through a complete solution of the problem. Suppose we have given a higher order characteristic equation, as, e.g.

$$a_0 p^n + a_1 p^{n-1} + a_2 p^{n-2} + \cdots + a_{n-2} p^2 + a_{n-1} p + a_n = 0 \quad (103)$$

In accordance with Routh† we arrange the coefficients in two rows

a_0	a_2	a_4	a_6
a_1	a_3	a_5	a_7

* E. A. Guillemin, *The Mathematics of Circuit Analysis*, chapter VI, article 26, Wiley, New York, 1949. See also F. E. Bothwell, "Nyquist Diagrams and the Routh-Hurwitz Stability Criterion," *Proc. I.R.E.*, **38**, 1345-1348 (1950) where an excellent historical survey is given.

† E. J. Routh, *Dynamics of a System of Rigid Bodies*, 3rd edition, p. 170, Macmillan, London, 1877. See also Gardner-Barnes,^{D8} p. 197; R. E. Doherty and E. G. Keller, *Mathematics of Modern Engineering*, p. 129, Wiley, New York, 1936; A. Porter, *An Introduction to Servomechanisms*, p. 48, Methuen's Monographs, Wiley, New York, 1950; and Truxal,^{B12} pp. 231, 267.

and form products of a_1 successively with each upper member and subtract the products of a_0 successively with each lower member so that we get the new coefficients

$$\begin{aligned} b_1 &= \frac{1}{a_1} (a_1 a_2 - a_0 a_3) \\ b_2 &= \frac{1}{a_1} (a_1 a_4 - a_0 a_5) \\ b_3 &= \frac{1}{a_1} (a_1 a_6 - a_0 a_7) \cdot \cdot \cdot \end{aligned} \quad (104)$$

Now form two new lines of coefficients, repeating the second line of the original set and adding below it the new set

$$\begin{array}{cccc} a_1 & a_3 & a_5 & a_7 \\ b_1 & b_2 & b_3 & b_4 \end{array}$$

and proceed exactly as before, i.e., form new coefficients by forming products of b_1 successively with each upper member and subtract the products of a_1 successively with each lower member

$$\begin{aligned} c_1 &= \frac{1}{b_1} (b_1 a_3 - a_1 b_2) \\ c_2 &= \frac{1}{b_1} (b_1 a_5 - a_1 b_3) \cdot \cdot \cdot \end{aligned}$$

This needs to be carried on until no further coefficients can be formed, because the rows have shrunk to one term each.

A necessary and sufficient condition that no roots with positive real parts exist is the fact that all terms in the first column of these coefficients have positive sign. Actually, the number of times the sign of the coefficients in the first column changes gives the number of roots with positive real parts. Since only the change of sign is of significance, one can always divide through the whole row of coefficients by a suitable positive number to keep the coefficients in the neighborhood of unity.

As an example let us take the fifth-order equation

$$(p+4)(p^2-4p+6)(p^2-p+5) = p^5 - p^4 + 5p^3 + 34p^2 - 74p + 120 = 0$$

Arranging the coefficients as required, we have

$$\begin{array}{ccc} 1 & 5 & -74 \\ -1 & 34 & 120 \end{array}$$

We form the new coefficients

$$b_1 = +39, \quad b_2 = +46$$

so that we now have, after dividing the b coefficients by 39

$$\begin{array}{ccc} -1 & 34 & 120 \\ +1 & +1.18 & \end{array}$$

The new coefficients are

$$c_1 = 35.18, \quad c_2 = 120$$

If we divide again by 35.18, we now are left with

$$\begin{array}{cc} 1 & 1.18 \\ 1 & 3.42 \end{array}$$

so that the only one new coefficient is possible

$$d_1 = -2.24$$

This leads to the triplet of numbers

$$\begin{array}{cc} 1 & 3.42 \\ -1 & \end{array}$$

so that the very last coefficient will be +3.42. Repeating the coefficients of the first column only, we now have

$$+1, -1, +1, +1, -1, +3.42$$

This indicates four changes of sign, which is in accordance with our original assumption.

This method gives only the occurrence of a positive real part of a root but it gives it correctly whether the root is real or complex. In order to get the actual root values, one can use any one of the methods briefly described in Appendix 3 of Vol. 1.

Should any term in the first column be zero, then the method will not be applicable, as is obvious from the law of formation of coefficients in (104). In such a case one might either assume a small but finite value instead of zero for the coefficient, though the sign might be uncertain, or introduce a change of variable by setting $p = q^{-1}$ which inverts the series of coefficients and transforms the root values into their reciprocals.

The condition of stability that the rational Laplace transform have no pole in the right-half p -plane can be formulated a little differently.*

* H. Nyquist, "Regeneration Theory," *Bell System Tech. J.*, **11**, 126-147 (1932).

Let us define the general transfer function of a network with a single feedback loop as

$$T(p) = \frac{A(p)}{1 - A(p)\beta(p)} \quad (105)$$

where $A(p)$ might be the voltage or current amplification factor and $\beta(p)$ the transfer function of the feedback network. The function $T(p)$ will in general be a rational function as long as we employ only lumped-parameter networks. An extension to transmission lines is, of course, quite simple. We see readily that in the form of (105) the function $T(p)$ has poles for all values p_α which make $A(p)\beta(p) = 1$.

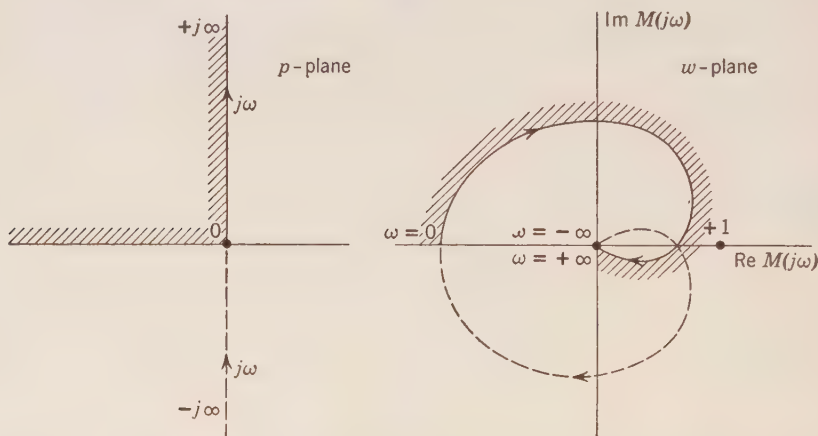


Fig. 5.22. Conformal mapping of imaginary axis of p -plane, $p = j\omega$, into w -plane, $w = A(p)\beta(p)$; Nyquist criterion.

If these poles are all in the left-half p -plane, we consider $T(p)$ the transform of a stable network.

We can, however, conformally map the left-half p -plane into the w -plane where the new complex variable is a function of p

$$w = A(p)\beta(p) = M(p)$$

The imaginary axis $p = j\omega$ will thus transform into a contour $w(j\omega)$ in the w -plane as shown in Fig. 5.22. As we let ω vary from $-\infty$ to $+\infty$ in the p -plane, we keep the left-half p -plane $\text{Re } p < 0$ to the left. If the image of this contour $p = j\omega$ in the w -plane lies entirely "to the right" of $w = 1$, then we have assurance that the "mapped region" which is to the left of the contour $w(j\omega)$ contains the point $w = 1$. This means in turn that all the p_α values which make $A(p_\alpha)\beta(p_\alpha) = 1$ belong into the mapped region, i.e., lie in the left-half p -plane. We

therefore need only plot the contour $w(j\omega)$ and observe its circulation with respect to the point $w = 1$; if the contour, progressing from $\omega = -\infty$ to $\omega = +\infty$, does *not* encircle $w = 1$ then the system is stable. Thus, we need not carry through the mapping process in detail, which might be quite troublesome.

Obviously this method can be extended to any type of functional relation, irrational and transcendental, if the precautions discussed in the later chapters on transmission lines are taken.

It should be pointed out that we have written (105) with the conventional negative sign in the denominator, as is customary in feedback amplifier theory. If we had chosen a positive sign as in (102), then the critical point would have to be taken as $w = -1$ and not $w = +1$ but the argument remains the same. This difference of sign is often apparent between authors dealing with feedback amplifier problems and those dealing with automatic feedback control systems and servo systems. The references in Appendix 3, section B, need therefore be examined for their point of view when problems of stability are being discussed.

PROBLEMS

5.1 Find the output voltage of the triode amplifier in Fig. 5.6, replacing G_L by the parallel combination of capacitance C_L and load G_L and applying step voltage $V_m I$ to the input. For numerical computation assume $C_L/G_L = 10^{-4}$ sec and disregard the interelectrode capacitances of the tube.

5.2 The pentode of Fig. 5.2 is connected to the series-peaked interstage network of Fig. 5.7b. Find the output voltage for a square-wave voltage pulse applied to the input. Select $Q = R^{-1} \sqrt{L/C} = 0.5$ and all other quantities as for 36a.

5.3 Carry through the analysis of the two-stage resistance-coupled amplifier (Fig. 31 in Arguimbau,* p. 159) for a square-wave voltage pulse applied to the input.

5.4 Carry through the analysis of the two-stage triode amplifier employing a series-peaked coupling network and having simple resistive load at the output. Apply a step voltage $V_m I$ at the input. Simplify the result by disregarding the interelectrode capacitances.

5.5 Evaluate the output voltage for the preceding problem for a single sawtooth voltage pulse. Disregard the interelectrode capacitances of the tubes.

5.6 Find the output of a two-stage triode amplifier with shunt-peaked interstage coupling and resistive load in the output. Apply to the input a square-wave voltage pulse.

5.7 The pentode of Fig. 5.2 is connected to a single section of a low-pass filter. Find the output voltage if a step voltage $V_m I$ is applied at the input.

5.8 A cascade of three stages of pentode amplifiers has two identical interstage series-peaked coupling networks and a resistive load in the output. Find the output voltage if a square-wave voltage pulse is applied at the input.

* L. B. Arguimbau, *Vacuum-Tube Circuits*, Wiley, New York, 1948.

5.9 In the same arrangement as in problem 5.8 assume that the two series-peaked interstage networks have $Q = 0.8$ and $Q = 0.63$, respectively. Find the output voltage if a square-wave pulse is applied at the input.

5.10 In the same arrangement as in problem 5.8 take as interstage coupling networks two identical capacitive sections as in Fig. 5.6c. Find the output voltage for a linearly rising input voltage.

5.11 A two-stage pentode amplifier has a band-pass filter of the type shown in Fig. 5.23 as coupling network. With a resistive load in the output, find the response to a step voltage applied at the input.

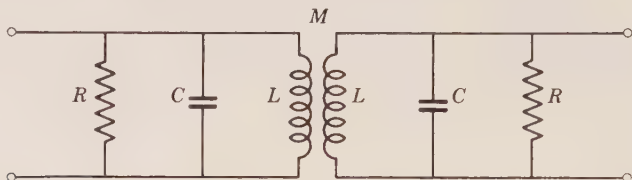


Fig. 5.23. Band-pass filter as coupling network for two-stage amplifier.

5.12 Extend the solution of problem 5.8 to N complete stages. Use the active fourpole line concept.

5.13 Construct the transistor dual to (a) the cathode follower circuit; (b) the grounded-grid triode.

5.14 Take the idealized transistor equivalent circuit of Fig. 5.16 with the capacitive coupling network of Fig. 5.7c. Apply a square-wave current pulse and find the output current.

5.15 Use the complete equivalent circuit, Fig. 5.15, for the transistor and combine it with the capacitive coupling network, Fig. 5.7c. (a) Apply a square-wave current pulse and find the output current. (b) Compare with the solution in the previous problem.

5.16 A chain of three transistor amplifiers uses two identical coupling networks as shown in Fig. 5.16. The output is terminated into a resistance R_L . Apply a unit step current at the input and find the response of the chain.

5.17 Construct a simple dual to the triode amplifier with coupling network, Fig. 5.7a, and interpret it as a transistor amplifier circuit.

5.18 Apply a sinusoidal voltage $V_m \sin \omega t$ to the feedback amplifier in Fig. 5.18 and find the output voltage.

5.19 Replace in the simple feedback circuit of Fig. 5.18 the shunt conductance G_1 by a shunt capacitance C . Apply a step voltage $V_m I$ at the input and find the output voltage. Check the stability limits of the circuit.

5.20 In the feedback circuit in Fig. 5.20 omit the capacitances in the forward section of the network but add to the feedback resistance r a shunt capacitance C . Find the response for a step voltage $V_m I$ applied at the input terminals.

Transmission Lines

The propagation of signals along a uniform conductor of relatively small cross section and very great length was initially studied in connection with telegraphy by W. Thomson* (Lord Kelvin) who only considered resistance and capacitance for a single conductor and thus did not establish true propagation effects. Kirchhoff† took into account the inductance, but also treated only a single conductor; he established a velocity of propagation close to that of light, but did not carry the investigations further. It was left for Heaviside‡ to apply Maxwell's theory to *guided propagation* along wires and to formulate the concept of "distortionless transmission;" however, even Heaviside considered only the principal wave and did not realize the full complexity of the electromagnetic field propagation along wires. The first complete solution for two parallel wires was given by Mie§ who solved the free equations within as well as outside the round solid conductors and established the existence of an infinity of transmission modes. Of course, it is not necessary, and we have no intention here, to present the *complete* solution of guided wave transmission|| in order

* W. Thomson, "On the Theory of the Electric Telegraph," *Math. Phys. Papers*, **2**, 61; reported in *Proc. Roy. Soc.*, May 1855, and published in *Phil. Mag.* (4), **11**, 146-160 (1856).

† G. Kirchhoff, "On the Motion of Electricity in Wires" [German], *Pogg. Ann.*, **100**, 193-217 (1857).

‡ O. Heaviside, *Electromagnetic Theory*; Vol. I, 1893, Vol. II, 1899; reprinted by Dover Publications, New York, 1950.

§ G. Mie, "Electric Waves on Two Parallel Wires" [German], *Ann. Physik*, **2**, 201-249 (1900). For further references see E. Weber, "Historical Notes on Microwaves," *Proc. Symposium on Modern Advances in Microwave Techniques*, Polytechnic Institute of Brooklyn (1954).

|| For a rigorous theory of transmission lines based upon electromagnetic field distributions, see R. W. King, *Transmission Line Theory*, McGraw-Hill, New York, 1955.

to solve the transmission-line problems of engineering importance in signal transmission. It is necessary, however, to understand clearly the rigorous background in order to appreciate the inherent limitations—along with the unparalleled elegance of the conventional transmission-line theory based upon the original *telegrapher's equation*, as the basic partial differential equation has been frequently called in mathematical treatises.

To be sure, the theory of the *wave equation* as well as the *equation of diffusion* of heat had been well developed at the time of the inception of telegraphy. The real complexity of the electrical propagation effects, however, could not be comprehended without the full benefit of Maxwell's field theory. It is therefore imperative to include a critical study of the *transmission-line concept* before entering upon the treatment of power and signal transmission with their respective demands upon line performance.

6. LOSSLESS AND DISTORTIONLESS TRANSMISSION LINES

In order to proceed from the simple to the more involved, we will, after establishing the transmission-line concept, first concentrate upon lossless and distortionless transmission lines which in their ideal forms are the ideal objectives of power and communication transmission, respectively. Certainly, power stations intend to transmit their electric power to the consumer with a minimum of loss, being most concerned with efficiency of the transmission. Certainly also, communication systems try to convey the information with a minimum of distortion, being most concerned with the fidelity of the transmission, even at the expense of high power loss necessitating repeater stations to restore a sufficient power level.

Though admittedly neither of the two objectives is completely met, the study of these two ideal transmission-line types will give the basic aspects with sufficient accuracy to warrant a detailed treatment.

6.1 The Transmission-Line Concept

We can approach the conventional formulation of transmission-line problems from two different backgrounds: we can either extrapolate from the lumped-fourpole line to elements extended in space, or we can start from the field concept in three-dimensional space and reduce it to the treatment of wave propagation in one dimension only. In both transitions we shall arrive at the same final result but with different insight into the real meaning of the conventional theory.

Let us start with the derivation of the smooth line equation from the fourpole line relations, which we treated in section 3.1. The process is quite similar to the transition from a system of vibrating individual mass points to the vibrating string.* We established the difference equation for three successive section entrance currents in their Laplace transform notations in (3.5)

* Lord Rayleigh, *Theory of Sound*, chapter VI; Dover Publications, New York, 1945.

$$I_{q-1}(p) - (\mathfrak{A} + \mathfrak{D})I_q(p) + I_{q+1}(p) = 0 \quad (1)$$

whereby the general parameters for a symmetrical T-section are defined as

$$\mathfrak{A} = \mathfrak{D} = 1 + YZ, \quad \mathfrak{B} = Z(2 + YZ), \quad \mathfrak{C} = Y$$

with Y the shunt and Z the series element of the ladder network. Applying this to a two-wire transmission line, we can select a short

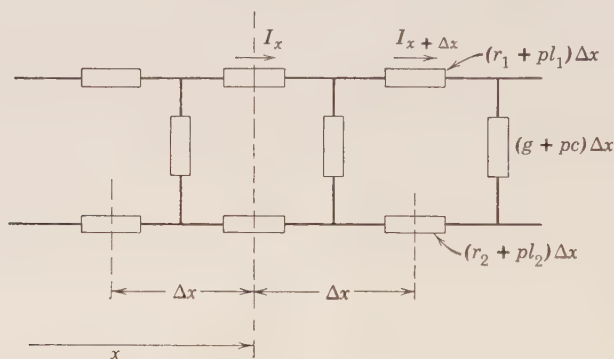


Fig. 6.1. Transition from fourpole line to smooth transmission line.

element Δx as in Fig. 6.1 and conceive of it as a symmetrical T-section with the elements

$$Y = (g + pc)\Delta x, \quad Z = (r + pl)\Delta x$$

r, l, g, c being respectively the resistance, inductance, leakance, and capacitance per unit length of the line. If we differentiate the two conductors by numbers 1 and 2, then $r = r_1 + r_2$, $l = l_1 + l_2$, the total resistance and inductance of go and return paths. We need for (1) the coefficient

$$(\mathfrak{A} + \mathfrak{D}) = 2[1 + (g + pc)(r + pl)(\Delta x)^2]$$

We had defined I_q as the transform of the current entering section q so that we now define its equivalent as $I_x(p)$, the transform of the current at the distance x from the sending end of the line and entering the element Δx . By Taylor's series expansion, if we assume for the transmission currents continuity in x and therefore the same for their transforms, we can write for $I_{q\pm 1}(p)$ the equivalent terms

$$I_{x\pm\Delta x}(p) = I_x(p) \pm \frac{\partial I_x}{\partial x} \Delta x + \frac{\partial^2 I_x}{\partial x^2} \frac{(\Delta x)^2}{2} \pm \dots \quad (2)$$

so that (1) becomes, combining first and last terms by (2)

$$2I_x(p) + \frac{\partial^2 I_x}{\partial x^2} (\Delta x)^2 - 2[1 + (g + pc)(r + pl)(\Delta x)^2]I_x(p) = 0$$

or, finally, since $2I_x(p)$ cancel and $(\Delta x)^2$ can be omitted

$$\frac{\partial^2 I_x(p)}{\partial x^2} = (r + pl)(g + pc)I_x(p) \quad (3)$$

a second-order differential equation in x for the current transform, which obviously must be a function of both x and p , the latter behaving as a parameter substituting for time. Of course, we could have started with the time relations underlying (1), but the direct use of the Laplace transforms is certainly more convenient. To recognize both the continuous variable and the parameter p in the Laplace transform, we shall write it from now on $I(x, p)$.

For the voltage transform $V(x, p)$ we can find the identical differential equation (3) if we start from the difference equation (3.7); alternately, we can express it in terms of the current transform, if we note that the shunt element carries the difference of successive section currents, so that

$$YV(x, p) = I_x(p) - I_{x+\Delta x}(p) = -\frac{\partial I_x}{\partial x} \Delta x$$

or also

$$(g + pc)V(x, p) = -\frac{\partial}{\partial x} I(x, p) \quad (4)$$

Conversely, we see that the current $I(x, p)$ can be related to the difference of the successive node-pair voltage transforms, namely

$$ZI(x, p) = V_x(p) - V_{x+\Delta x}(p) = -\frac{\partial V_x}{\partial x} \Delta x$$

or

$$(r + pl)I(x, p) = -\frac{\partial}{\partial x} V(x, p) \quad (5)$$

We must recall that the difference equations of the fourpole line were derived for de-energized initial conditions, and this holds equally for (4) and (5), as well as (3). In addition, (4) and (5), coming by transition from the lumped-circuit realm, are subject to the same limitations pointed out in establishing the circuit concept in Vol. I section 1.1, namely: it is assumed that the electromagnetic fields associated with currents and potentials at any one point are established instantaneously throughout space, and that radiation effects are therefore dis-

regarded, even though the establishment of current and voltage along the line proceeds with the appropriate velocity of propagation. This means, in turn, that the fields can only be considered reasonably well defined near the transmission-line wires and are not even approximately correct at large lateral distances from them. To be a little more quantitative, Guillemin,^{A8} Vol. II, pp. 12–24, has attempted a semirigorous establishment of the nature of this approximation for steady-state conditions. For transient conditions, we cannot accept much of this justification.

Now let us briefly describe the rigorous approach and set down the simplifications needed to fit in with conventional transmission-line theory. Fortunately, time has passed since the early concern about the accuracy of the transmission-line concept was voiced and the advent of wave-guide theory has made us familiar with many ideas then thought to be rather abstract. Assume at first a parallel plate transmission system, rather than a two-wire transmission line, as indicated in Fig. 6.2. The source of energy is provided by the antenna $A'A''$, where the gap represents the actual source of electromagnetic radiation. This antenna feeds current into one of the parallel plates and at some distance $x' = D$ in Fig. 6.2*b* the current will have reached a final cross-sectional distribution in this plate which we might consider as remaining the same from there on; usually this distance D is several times the spacing between the plates. By symmetry, the returning current in the opposite plate will show the same distribution pattern. At some rather far distance, the current will be utilized in a load, but we shall not discuss this in detail because it will be quite similar in principle to the situation at the “sending end.”

The actual field distribution within the conductor plates as well as in the space between them can be found only by a rigorous solution of Maxwell's field equations satisfying all the specified boundary conditions. As in the much simpler classical circuit analysis, the first step is a search for all possible “modes” of solution; that means partial solutions which satisfy the simplest boundary conditions and represent waves propagating (or exponentially decaying) in the x -direction. There are three types of such solutions, all called *plane waves*: (1) A completely transverse mode, i.e., electric and magnetic field vectors lying entirely in a plane normal to the two plates; this is the *principal mode*, with a field distribution as indicated in Fig. 6.2*c*. (2) An infinity of modes with the magnetic field vector completely transverse, but with the electric field vector having in addition to the transverse a slight longitudinal component in the propagation direction x ; these are the *transverse magnetic (T.M.) modes*. (3) An infinity of modes with

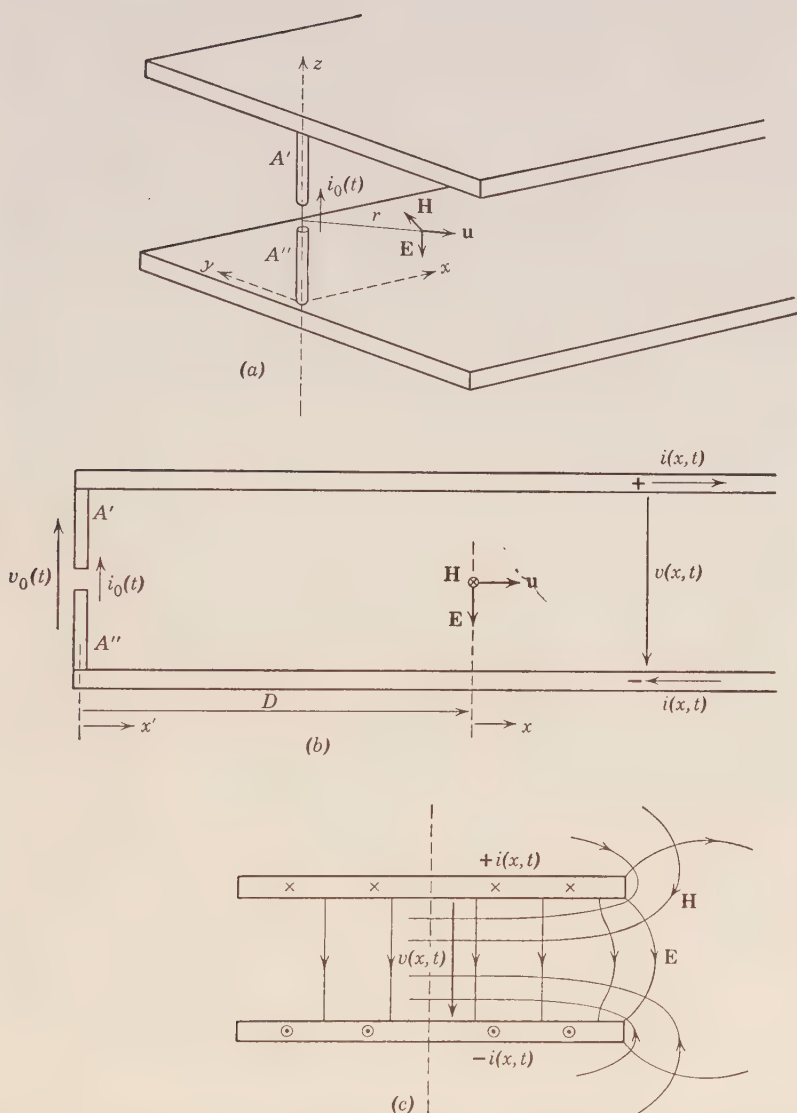


Fig. 6.2. Parallel plate transmission system. (a) Physical arrangement, (b) lateral view, (c) longitudinal view at $x > 0$ with approximate principal mode field pattern.

the electric field vector completely transverse, but with the magnetic field vector having in addition to the transverse a slight longitudinal component; these are the *transverse electric (TE) modes*.

The propagation characteristics of these various modes differ widely. The *principal mode* propagates with the velocity of light and with no attenuation; its electric field vector is normal to the metallic boundaries and cannot produce any losses. The *TM* and *TE modes*, on the other hand, propagate with velocities which depend on the geometry and the order number of the mode; these modes are generally attenuated at rates which normally increase with the order number.

If the conductors had infinite conductivity and the dielectric no losses, the "uniform" field section $x > 0$ would behave exactly like an ideal high-pass filter. The principal mode and certain complementary modes would propagate without attenuation; all other complementary modes would not propagate at all. The finite conductivity of the conductors will not affect the principal mode but will add attenuation to all the complementary modes; the electric field components in the propagation direction of the TM modes must be continuous on the metal surface and their sum corresponds there to the surface current density multiplied by the resistivity of the metal. All the higher TM modes as well as all of the TE modes attenuate very rapidly, and for all practical purposes exist only within the region $x' < D$, close to the "source" region. They have a great variety of field distribution patterns and by proper superposition can be made to match the complicated field pattern in a plane $x' > 0$ near the antenna. The evaluation of the required amplitudes, akin to the Fourier series analysis in many respects, is a rather tedious task.

We also observe that the principal mode has a Poynting vector entirely parallel to the transmission plates; it carries its energy without loss from the input plane of the "uniform" field section at $x = 0$ to the corresponding output plane at the load end; it cannot account for any lateral radiation from the transmission system. The propagating TM modes, on the other hand, because of their longitudinal electric field components, possess a lateral component of their Poynting vectors, but directed into the conducting plates; they account for the loss of energy (attenuation) by dissipation within the conductors but do not radiate energy into space. It must remain, then, for the modes existing in the "source" region to account for any radiation of energy into space.* Indeed, if we examine the field distribution close to the

* J. R. Carson, "The Guided and Radiated Energy in Wire Transmission," *Trans. AIEE*, **43**, 908 (1924); "Wire Transmission Theory," *Bell System Tech. J.*, **7**, 268-280 (1928).

antenna simulating the power source, we find magnetic field lines closed in horizontal planes which lead to a Poynting vector in a radial direction as indicated in Fig. 6.2a.

It is now obvious that the genuine principal mode of the rigorous theory cannot possibly describe the actual propagation of waves along dissipative conductors. Rather than complicate matters by the intricate mode structure of the rigorous theory, we introduce a *quasi-principal mode* or *principal wave* composed of the rigorous principal mode with slight longitudinal electric and magnetic field components to account for the losses in the guiding conductors and the separating dielectric. The advantages of this compromise are quite evident: we can compute the transverse electric and magnetic field components exactly as in the principle mode, i.e., as rigorous solutions of two-dimensional potential problems assuming the conductors to be ideally or infinitely conductive; we add an electric longitudinal component exactly identical with the tangential electric field on the surface of the conductor and determined by its conductivity and the local current density; and we add a slight magnetic longitudinal component to account for the transverse leakage current. This principal wave is now *quasi-plane* or *quasi-transverse*; it is attenuated along the parallel conductor system at a rate defined by the geometry and material of the conductors and dielectrics, and its associated Poynting vector has one component in the direction of energy transmission and another component normal to the conductor surfaces to account for the dissipation. Obviously, the assumptions made cover only the electric wave propagation in the uniform part of transmission lines, far enough from their terminals to permit substitution of this simplified picture. For the study of power transmission or signal propagation along wires, this procedure has given satisfactory results in most practical transmission systems which are constructed with reasonable efficiency; the fact remains, however, that this manner of describing the propagation phenomenon is only approximate and certainly completely invalid near the ends of the transmission system.

6.2 The General Transmission-Line Equations

A number of attempts have been made, starting with the field equations, to deduce the transmission-line equations* as first-order approximations for guided wave propagation. These involve, how-

* J. R. Carson, "Electromagnetic Theory and the Foundations of Electric Circuit Theory," *Bell System Tech. J.*, **6**, 1-17 (1927); L. A. Pipes, "An Operational Treatment of Electromagnetic Waves Along Wires," *J. Appl. Phys.*, **12**, 800-810 (1941).

$$\oint E_s ds = - \frac{\partial \Phi_m}{\partial t}, \quad \oint H_s ds = I + \frac{\partial \Psi_e}{\partial t} \quad (6)$$

where the line integrals of the electric field strength \mathbf{E} and the magnetizing force \mathbf{H} are to be extended over closed paths; Φ_m is the total magnetic flux through the one, Ψ_e the total dielectric flux through the other closed path or loop (each counted positive if, looking in the direction of its flow, the line integral appears clockwise); and I is the conduction current penetrating the magnetic loop.

Let us select the path for the electric line integral in the plane through the axes of the conductors as marked a, b, c, d in Fig. 6.3*b*. Since the conductors are ideal and have infinite conductivity, their surfaces in any one cross section will have definite "potentials," say Φ_1 and Φ_2 , so that from electrostatics, with $\Phi_1 - \Phi_2 = v$, the voltage

$$\int_a^b E_s ds = (\Phi_1 - \Phi_2)_{x+\Delta x} = v_{x+\Delta x} = v_x + \frac{\partial v_x}{\partial x} \Delta x$$

if we immediately apply Taylor's expansion to approximate the voltage value at $x + \Delta x$. The paths bc and da cannot give any contribution because the electric field vector is normal to the wire surfaces. The last section of the path gives

$$\int_c^d E_s ds = (\Phi_2 - \Phi_1)_x = -v_x$$

The total value of the closed line integral is now

$$\oint E_s ds = \frac{\partial v_x}{\partial x} \Delta x = - \frac{\partial}{\partial t} \Phi_m = - \frac{\partial}{\partial t} (l_e \Delta x i_x)$$

where we defined the magnetic flux in terms of the local average per unit length $l_e i_x$ with l_e the *external inductance* per unit length. Because of the infinite conductivity, the current will flow entirely in the surface of the wire and no internal magnetic field will exist. We thus have as final relation, putting in evidence time and space variables

$$\frac{\partial}{\partial x} v(x, t) = -l_e \frac{\partial}{\partial t} i(x, t) \quad (7)$$

The path for the magnetic line integral in (6) is perhaps best selected in the center plane between the conductor marked A, B, C, D in Fig. 6.3. Since no conductance current penetrates the loop, we are only concerned with the dielectric flux. Again, the horizontal section BC and DA will not contribute, because the magnetic field vector lies

completely within a cross-sectional plane. We can do no better than just write

$$\int_A^B H_s ds + \int_C^D H_s ds = \frac{\partial}{\partial t} \int_D^C (D_x \Delta x) ds$$

where we approximate the dielectric flux by the integral of the flux density D_x at x over the path length DC and multiply by Δx to get the area of the loop. Here it is that we must make one of the important concessions. If the path AB is extended to infinity, then the integral can be considered the equivalent of any closed line integral along a magnetic field line, say S_1 in Fig. 6.3a, and the value would be $-i_{x+\Delta x}$, that for the correspondingly extended integral along CD would be $+i_x$, and the total dielectric flux would equal the charge $q_x \Delta x$ on the length Δx of conductor 1 from which the dielectric flux emanates. Because of the electrostatic character of the dielectric field, we can also use $q_x = cv_x$ with c the *capacitance* per unit length. We may thus write

$$\oint H_s ds = -\frac{\partial i_x}{\partial x} \Delta x = +\frac{\partial}{\partial t} \Psi_e = +\frac{\partial}{\partial t} (c \Delta x v_x)$$

from which we deduce, again indicating clearly time and space variables

$$\frac{\partial}{\partial x} i(x,t) = -c \frac{\partial}{\partial t} v(x,t) \quad (8)$$

This second line equation is strictly valid only near the conductors, because only there can we relate the magnetic line integral to total current; far from the conductors we could only do this if we were to assume infinite velocity of propagation so that we could disregard any phase difference between the field close to and far from the conductors. This, of course, is identical with disregarding radiation effects!

We now have the pair of transmission-line equations (7) and (8) for the ideal conductors with ideal dielectric. If there are slight losses, we can make the following amendments, in keeping with the concept of the quasi-principal mode as defined in the previous section: (1) We admit in the line integral of the electric field vector slight longitudinal contributions which in terms of voltage change $(\partial v/\partial x)$ mean the resistive drop ri in both conductors, $r = r_1 + r_2$ if the conductors have individually different resistances per unit length. (2) Since the current now distributes over the conductor cross sections, we extend the line integral of the electric field vectors to $a'b'c'd'$ as in

Fig. 6.3*b*, which are located on the kernel line* along which the magnetic field vanishes so as to include the entire magnetic linkage; we thus change l_e appropriately to l . (3) We admit in the line integral of the magnetic field vector slight longitudinal contributions which in terms of current change $(\partial i/\partial x)$ mean the leakage loss of current gv from conductor 1 to conductor 2.

The amended transmission line equations are now

$$-\frac{\partial}{\partial x} v(x,t) = ri(x,t) + l \frac{\partial}{\partial t} i(x,t)$$
$$-\frac{\partial}{\partial x} i(x,t) = gv(x,t) + c \frac{\partial}{\partial t} v(x,t)$$

(9)

and these show definite relationship to the transform equations (4) and (5). We observe that the amendments really come from the

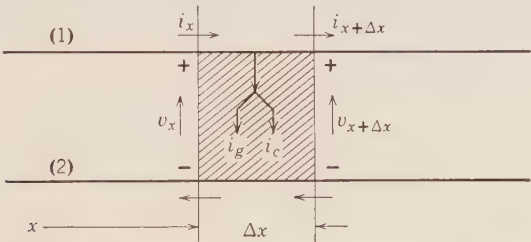


Fig. 6.4. Kirchhoff's circuit equations applied to the transmission line.

conductor medium. We retain outside in the dielectric the transverse nature of the field distribution, so that the capacitance and external inductance values are still the static values found from potential theory and we admit small discrepancies in the boundary values of the tangential field components on the conductor surfaces.

There is still another way, also a compromise, for deducing the transmission-line equations, frequently chosen in textbooks as the most plausible one. It uses directly the Kirchhoff equations for short elemental lengths of the conductors which are assumed of very small cross sections so that one can avoid subsidiary integrations. Referring to Fig. 6.4 we apply Kirchhoff's first law, that the sum of all active electromotive forces around a closed loop be equal to the resistive drop. For our purpose here we choose to call the line voltages v_x and $v_{x+\Delta x}$

* E. Weber, *Electromagnetic Fields*, Vol. I—*Mapping of Fields*, p. 154, Wiley, New York, 1950.

“impressed electromotive forces” since they are, indeed, maintained by the energy source. Going in the clockwise direction around the shaded rectangle we have then

$$(r_1 + r_2)i_{av} = -v_{x+\Delta x} + v_x - \frac{\partial \Phi_m}{\partial t} \quad (10)$$

with the last term representing the electromotive force of self-induction. Taking the Taylor series approximations

$$i_{x+\Delta x} = i_x + \frac{\partial i_x}{\partial x} \Delta x, \quad v_{x+\Delta x} = v_x + \frac{\partial v_x}{\partial x} \Delta x$$

the average value of the current over element Δx is, e.g.

$$i_{av} = \frac{1}{2} (i_x + i_{x+\Delta x}) = i_x + \frac{\Delta x}{2} \frac{\partial i_x}{\partial x} \quad (11)$$

As we let $\Delta x \rightarrow 0$, the second term can be discarded unless the rate of change of current is enormous. The magnetic flux in the elemental rectangle will be with (11)

$$\Phi_m = l \Delta x i_{av} \approx l \Delta x i_x$$

where l is the total inductance per unit length. Introducing all the pertinent expressions, (10) reduces to

$$ri_x + l \frac{\partial i_x}{\partial t} = - \frac{\partial v_x}{\partial x} \quad (12)$$

which is at once the first line of (9). Since the Kirchhoff equations are based on the circuit concept, we have obviously included all the pertinent restrictions but have cast aside any indications of these so that we seem to have given a logical and rigorous derivation of this transmission-line equation. It is for this reason that we have devoted considerable space to emphasizing the limitations of the conventional theory.

The second law of Kirchhoff states that the sum total of all currents at any junction must be zero. If we take the element Δx as a current junction (eventually $\Delta x \rightarrow 0$), we can read off Fig. 6.4 directly

$$i_x = i_{x+\Delta x} + i_c + i_g \quad (13)$$

Actually, the currents i_c and i_g are uniformly distributed along the conductor and are defined by

$$i_c = \frac{\partial q}{\partial t} = c \Delta x \left(\frac{\partial v}{\partial t} \right)_{av}, \quad i_g = g \Delta x v_{av}$$

with c and g again designating capacitance and leakage per unit length. The average value of the voltage can be taken as v_x in analogous manner to (11) and we make the same assumption about $(\partial v/\partial t)$. Thus, (13) becomes

$$-\frac{\partial i_x}{\partial x} = gv_x + c \frac{\partial v_x}{\partial t} \quad (14)$$

which is the second line of (9).

Combining (12) and (14), e.g., by differentiating (12) with respect to x , substituting (14) for $(\partial i)/(\partial x)$, and taking for $(\partial^2 i_x)/(\partial x \partial t)$ again (14) after differentiating it with respect to time, we obtain the *telegrapher's equation* for the voltage

$$\frac{\partial^2 v_x}{\partial x^2} = lc \frac{\partial^2 v_x}{\partial t^2} + (lg + cr) \frac{\partial v_x}{\partial t} + rgv_x \quad (15)$$

If there are no losses, this reduces to the *ideal wave equation*

$$\frac{\partial^2 v_x}{\partial x^2} = lc \frac{\partial^2 v_x}{\partial t^2} \quad (16)$$

which had been established very early for the displacement of a vibrating string* and treated by d'Alembert (1747), Fourier (1807), and Cauchy (1823), each by different methods.

6.3 The General Solution; Distortionless Line Concept

The classical method of solution proceeds from the *telegrapher's equation* (15), which, in fact, is a wave equation amended to account for the losses. It represents a partial differential equation with constant coefficients in two variables, the time t which has infinite range of variability and the space coordinate x which might either have an infinite or finite range of variability. It has become customary to designate the specific values which the solution has to satisfy as $t = 0$ as *initial conditions* and the specific values it has to satisfy at one or both ends of its physical extension as *boundary conditions*. All the lumped-network problems involved time only and belong to the class of *initial-value problems*; the fourpole line, although involving only time as a continuous variable, led to the distinction of sending and receiving ends and therefore was properly a forerunner to the transmission-line problem, which belongs distinctly to the group of *boundary-*

* A. G. Webster, *Partial Differential Equations of Mathematical Physics*, G. E. Stechert, New York, 1933; R. V. Churchill, *Fourier Series and Boundary Value Problems*, p. 21, McGraw-Hill, New York, 1941.

value problems, particularly to the *one-dimensional* boundary value problems, because of the single space variable.

As the most general type of wave equation with constant coefficients, the telegrapher's equation has been the object of much thorough examination as to possible kinds of solutions, existence and convergence of solutions, and compatibility of boundary conditions. The critical examination from the mathematical point of view is found in many of the texts dealing with partial differential equations of mathematical physics. Perhaps one of the most searching is by Courant and Hilbert;* a very comprehensive treatise is by Bateman.†

We are concerned here less with the general mathematical details and more with the application of the Laplace transform method since it presents unusual advantages over any other method of solution and is particularly valuable for engineering applications. It should be understood that the Heaviside-Jeffreys operational method also is generally more efficient than the classical methods of solution; however, many results can be established rigorously by Laplace transforms whereas the strictly operational approach might leave room for doubt. Nevertheless, credit must be given to Heaviside‡ for the development of many original contributions which have brought transmission-line theory much farther along than corresponding areas of problems in other fields of physics.

To introduce the Laplace transform formulation, we start best with the space-time differential equations (9), to wit

$$\begin{aligned} -\frac{\partial}{\partial x} v(x,t) &= ri(x,t) + l \frac{\partial}{\partial t} i(x,t) \\ -\frac{\partial}{\partial x} i(x,t) &= gv(x,t) + c \frac{\partial}{\partial t} v(x,t) \end{aligned} \tag{17}$$

Here, both x and t are continuous variables. The functions $i(x,t)$ and $v(x,t)$ must be assumed to be differentiable with respect to both variables except perhaps at certain isolated values of the variables similar to the simpler time functions like unit step.

We shall apply the transformation with respect to t as before, since there is no doubt as to its range of variability; in so doing we must

* R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, J. Springer, Berlin, Vol. I, 1931; Vol. II, 1938. Also, *Methods of Mathematical Physics*, Interscience Publishers, New York, 1953.

† H. Bateman, *Partial Differential Equations of Mathematical Physics*, Dover Publications, New York, 1944.

‡ *Loc. cit.*

consider x as a continuous real parameter. Let us define, then

$$\mathcal{L}i(x,t) = \int_{t=0^+}^{\infty} i(x,t)e^{-pt} dt = I(x,p) \quad (18)$$

and similarly for the voltage $v(x,t)$. For both current and voltage we also need the transforms of their first partial derivatives. If $i(x,t)$ possesses a continuous partial derivative with respect to x , and if $I(x,p)$ is a continuous function of x , then we certainly can write

$$\mathcal{L} \frac{\partial i(x,t)}{\partial x} = \int_{t=0^+}^{\infty} \frac{\partial i}{\partial x} e^{-pt} dt = \frac{\partial}{\partial x} I(x,p) \quad (19)$$

by simply interchanging integration and differentiation; of course, where either $i(x,t)$ or $I(x,p)$ or both possess a discontinuity with respect to x , we could not write (19). With x a continuous parameter in (18), we can readily find the transform of the time derivative by the use of Table 1.4, namely

$$\mathcal{L} \frac{\partial i(x,t)}{\partial t} = pI(x,p) - i(x,0) \quad (20)$$

where now $i(x,0)$ represents the *initial distribution* of the current over the transmission line which must be known. The same relations also hold, of course, for the voltage and its transforms. Introducing the appropriate expressions into (17) leads to the system

$$\begin{aligned} -\frac{\partial V(x,p)}{\partial x} &= (r + pl)I(x,p) - li(x,0) \\ -\frac{\partial I(x,p)}{\partial x} &= (g + pc)V(x,p) - cv(x,0) \end{aligned} \quad (21)$$

Actually, in spite of the partial derivative notation retained here, these are now two ordinary differential equations in the single variable x with p as a continuous, complex parameter and with inhomogeneous terms in x constituting the initial distributions of current and voltage, which must, indeed, be given as initial conditions of the problem. We can now verify that (21) is the same transform system as (4) and (5) which were directly obtained from the fourpole line difference equations! We are missing the initial functions $i(x,0)$ and $v(x,0)$ in (4) and (5) but this is natural since we had disregarded any initial energies in establishing them.

To effect a final solution, we can now proceed with the system (21) as with any system of ordinary first-order differential equations. Let us briefly write just V and I for the transforms, differentiate the first

line with respect to x , assuming that this can be done, i.e., that the required derivatives actually exist, and introduce the second line; then

$$\frac{d^2 V}{dx^2} = (r + pl)(g + pc)V + l \frac{d}{dx} i(x,0) - c(r + pl)v(x,0) \quad (22)$$

This must be the Laplace transform of the telegrapher's equation (15) containing the initial conditions as subsidiary terms. An entirely similar form is obtained if we differentiate the second line of (21) and introduce into it the first line, namely

$$\frac{d^2 I}{dx^2} = (r + pl)(g + pc)I + c \frac{d}{dx} v(x,0) - l(g + pc)i(x,0) \quad (23)$$

Either of these two inhomogeneous, ordinary second-order differential equations with constant coefficients can be solved straightforwardly by the methods discussed in Vol. I section 2.5. Actually, it is not necessary to solve both equations, since having the solution for one, say (22), we can utilize the first line (21) to obtain the current transform solution. Let us then concentrate on (22). For convenience we shall denote

$$n^2 = (r + pl)(g + pc) = \frac{1}{u^2} [(p + \delta)^2 - \sigma^2] \quad (24)$$

and we rewrite it by completing the square in p . The new constants are by comparison of the expanded forms

$$u = \frac{1}{\sqrt{lc}}, \quad \delta = \frac{1}{2} \left(\frac{r}{l} + \frac{g}{c} \right), \quad \sigma = \frac{1}{2} \left(\frac{r}{l} - \frac{g}{c} \right) \quad (25)$$

The basic homogeneous part of (22) is now simply

$$\frac{d^2 V}{dx^2} = n^2 V \quad (26)$$

and has the exponential function of argument $\pm nx$ as solution. The general solution is therefore

$$V(x,p) = Ae^{nx} + Be^{-nx} \quad (27)$$

The inhomogeneous terms in (22) add as particular integral

$$\frac{1}{2n} [e^{nx} \int e^{-nx} M_v(x) dx - e^{-nx} \int e^{nx} M_v(x) dx] \quad (28)$$

where we abbreviated

$$M_v(x) = l \frac{d}{dx} i(x, 0) - c(r + pl)v(x, 0)$$

From the first line of (21) we obtain for that part of the current transform solution corresponding to (27)

$$I(x, p) = -\frac{n}{r + pl} (Ae^{nx} - Be^{-nx}) = \frac{1}{Z_c} (-Ae^{nx} + Be^{-nx}) \quad (29)$$

We introduced in analogy to the fourpole line designation Z_c as the characteristic impedance with the definition

$$Z_c = \sqrt{\frac{r + pl}{g + pc}} = \sqrt{\frac{l}{c}} \sqrt{\frac{p + \delta + \sigma}{p + \delta - \sigma}} \quad (30)$$

The inhomogeneous part (28) contributes then in addition the terms

$$\frac{1}{2nZ_c} [-e^{nx} \int e^{-nx} M_v(x) dx - e^{-nx} \int e^{nx} M_v(x) dx] + \frac{li(x, 0)}{nZ_c} \quad (31)$$

Obviously, if we had chosen to solve (23) directly, the inhomogeneous part (31) would have appeared in different form but would have to lead to the same final result as long as the initial distribution functions are compatible, i.e., satisfy (9) for $t = 0$. As a matter of fact, from (9) we can take the equivalences for $t = 0$

$$\begin{aligned} \frac{d}{dx} i(x, 0) &= -gv(x, 0) - c \left(\frac{\partial v(x, t)}{\partial t} \right)_{t=0} \\ \frac{d}{dx} v(x, 0) &= -ri(x, 0) - l \left(\frac{\partial i(x, t)}{\partial t} \right)_{t=0} \end{aligned} \quad (32)$$

We observe that the solutions for the initially de-energized line as given by (27) and (29) have seemingly simple forms. Unfortunately, however, the exponent n as shown in (24) is an irrational function of p and, thinking of the task it is to find the inverse Laplace transform, this fact is not immediately inviting. If we scan the individual coefficients (25) we find $u = 1/\sqrt{lc}$ to have the physical dimension of velocity and we shall identify it as the finite, definite *velocity of propagation* of waves along the transmission line; u is actually independent of dissipation, depending only upon inductance and capacitance, much akin to the velocity of uniform plane electromagnetic waves in space for which $u = 1/\sqrt{\epsilon\mu}$ with ϵ the absolute dielectric constant and μ the absolute permeability.

The coefficient δ vanishes only if $r = g = 0$, i.e., if there are no losses; it will appropriately be identified as an *attenuation constant*. If $r = g = 0$ then σ also vanishes and

$$n_0 = p \sqrt{lc} = \frac{p}{u} \quad (33)$$

For the line without losses, the propagation function n reduces thus to a very simple rational form so it will be considered first.

The only other simplification can arise when

$$\frac{r}{l} = \frac{g}{c} = \delta, \quad \sigma = 0 \quad (34)$$

because then from (24)

$$n_d = \frac{p + \delta}{u} \quad (35)$$

and the propagation function again takes a simple rational form. This fact was recognized first by Heaviside* who defined σ as the *distortion constant*, and called a line for which $\sigma = 0$ a *distortionless line*. This will be the second most important special case which we shall consider in the next section, leaving the more intricate applications for the later sections. Obviously, the lossless line is also distortionless since for it $\delta = \sigma = 0$.

6.4 Traveling Waves on Infinite Distortionless Lines

Let us first consider the line with no losses, for which $n = p/u$. In the definition of the Laplace transform we have stipulated $\text{Re}(p) > 0$ in order to assure convergence of the infinite integrals. This means then that

$$\text{Re}(n_0) = \frac{1}{u} \text{Re}(p) > 0 \quad (36)$$

so that in (27) the positive exponential indicates increasing values with x , whereas the negative exponential means decreasing values. For the infinitely long line, only the second possibility represents a physical solution, so that from (27) and (29)

$$V(x, p) = B e^{-p \frac{x}{u}}, \quad I(x, p) = \frac{B}{Z_c} e^{-p \frac{x}{u}} \quad (37)$$

* *Op. cit.*, Vol. I, p. 413.

The characteristic impedance from (30) reduces for either the lossless, or the distortionless, condition to the real value

$$Z_{c0} = Z_{cd} = \sqrt{\frac{L}{C}} = R_s \quad (38)$$

which we shall call the *surge resistance* of the line.

We still need to determine the integration constant B which must follow from the boundary conditions. As shown in Fig. 6.5 we might apply at the sending end a voltage $v_0(t)$ over a terminal impedance

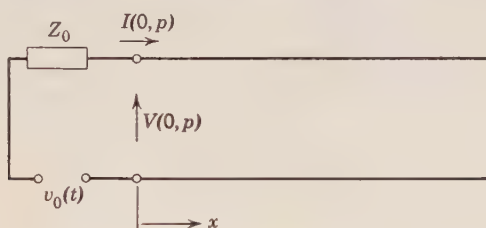


Fig. 6.5. Sending end termination of an infinitely long line.

$Z_0(p)$. The condition at the sending end can then be formulated by conventional circuit theory in terms of the Laplace transforms

$$V(0, p) = \mathcal{L}v_0(t) - Z_0 I(0, p) \quad (39)$$

where $V(0, p)$ and $I(0, p)$ are rather obviously the transmission-line transforms from (37) taken at $x = 0$, thus reducing to functions of p only, entirely compatible with the Laplace transform of the applied voltage which we have not yet specified. Using (37), we get from (39)

$$B = \mathcal{L}v_0(t) - \frac{Z_0}{Z_c} B$$

or

$$B = \frac{Z_c}{Z_c + Z_0} \mathcal{L}v_0(t) \quad (40)$$

The integration constant will therefore be a function of p , so that it always *must* be explicitly determined before the inverse Laplace transform for current or voltage can be taken.

As (38) shows, the characteristic impedance of the line is real. Dealing with a *lossless* line which is best approximated by a power transmission line, we shall certainly not want to introduce any unnecessary losses in the terminal equipment, so that we might best assume

$Z_0 = 0$. The voltage transform is then from (37) with (40)

$$V(x, p) = \mathcal{L}v_0(t)e^{-p\frac{x}{u}} \quad (41)$$

From Table 1.4 we find the inverse transform simply as a delayed exact reproduction of the applied voltage

$$v(x, t) = v_0\left(t - \frac{x}{u}\right) \quad t \geq \frac{x}{u} \quad (42)$$

The delay time $\tau = x/u$ is the exact time of travel with the uniform velocity $u = 1/\sqrt{lc}$. An oscillograph anywhere along the line would show exactly the same oscillogram as at the sending end but starting after an elapsed time interval τ proportional to the distance from the sending end. The current has exactly the same form as the voltage

$$i(x, t) = \frac{1}{\sqrt{l/c}} v_0\left(t - \frac{x}{u}\right) \quad t \geq \frac{x}{u} \quad (43)$$

For the distortionless line we take the propagation function n_d from (35), where now again

$$\operatorname{Re}(n_d) = \frac{1}{u} [\delta + \operatorname{Re}(p)] > 0$$

so that for the infinite line again only the second part of (27) and (28) is admissible. We now have

$$V(x, p) = B e^{-\delta\frac{x}{u}} e^{-p\frac{x}{u}}, \quad I(x, p) = \frac{B}{Z_c} e^{-\delta\frac{x}{u}} e^{-p\frac{x}{u}} \quad (44)$$

The first exponential factor signifies attenuation along the line; it is a factor independent of p and therefore need only be added in the final solution of the lossless line! We can thus accept the value of B from (40) and, if we again choose $Z_0 = 0$, we can even take the final solutions (42) and (43), add the attenuation factor, and have as a solution for the infinite distortionless line

$$\begin{aligned} v(x, t) &= e^{-\delta\frac{x}{u}} v_0\left(t - \frac{x}{u}\right) \\ i(x, t) &= \frac{1}{\sqrt{l/c}} e^{-\delta\frac{x}{u}} v_0\left(t - \frac{x}{u}\right) \end{aligned} \quad t \geq \frac{x}{u} \quad (45)$$

Here the local oscillogram would again be identical with the original except that it has shrunk in scale because of the local scale factor

$e^{-\delta(x/u)}$. If we picture the signal moving along the line, it will look somewhat distorted because of the spatial attenuation.

Assume the input signal to be a single sawtooth, of amplitude V_m and duration T , shown in Fig. 6.6a. At $x = 0$ the oscillogram will be the original sawtooth. At $x_1 = ut_1$, with $t_1 > T$, the first tip of the sawtooth will appear at exactly $t_1 = x_1/u$ and will rise linearly

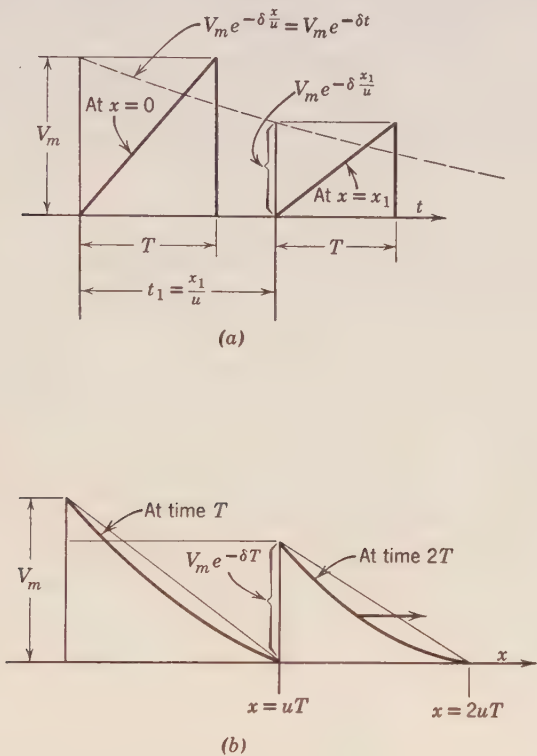


Fig. 6.6. Propagation of a sawtooth pulse along a distortionless line. (a) Oscillograms at $x = 0$, $x = ut_1$; (b) line distributions at $t = T$ and $t = 2T$.

to the maximum $V_m e^{-\delta(x/u)} = V_m e^{-\delta t_1}$ within the pulse time T . We can draw the oscillograms everywhere along the line by indicating the exponential $e^{-\delta t}$. The local attenuation factor is identical with the ordinate of this exponential above the local propagation time. On the other hand, the distribution of the pulse along the line is shown in Fig. 6.6b. At time $t = T$, the pulse has just emerged fully; its tip is at $x = uT$, its amplitude is at $x = 0$ and equal to V_m ; in between, the linear ordinates must be multiplied by the local attenuation factor $e^{-\delta(x/u)}$ so that it presents a curved rise and, of course, is mirrored as

compared with the oscillogram. At time $2T$, the tip has advanced to $x = 2uT$, the amplitude is located at $x = uT$ (since its duration is T) and has value $V_m e^{-\delta T}$ as we can take from the exponential in plot *a*. Again, of course, it has a curved rise, exactly proportional at each relative distance to the first drawn pulse.

The solutions discussed represent *traveling waves*, with genuine propagation characteristics. Because of the distortionless propagation, the lines satisfying $\sigma = 0$ are the most interesting special case. Unfortunately, this condition cannot easily be realized in practice as the many endeavors beginning with Heaviside have shown. The normal case for transmission lines is actually

$$\frac{r}{l} \gg \frac{g}{c}$$

To approach the distortionless condition one has several possibilities, all of which were tried at one time or another. Obviously, decrease of resistance r or increase of leakage g are both economically unsound, though Heaviside* demonstrated the effectiveness of the latter by actual experiment and showed the improvement of transmission quality. Decrease in capacitance c is very difficult to achieve beyond the normal separation of transmission-line wires in air, so that the only really feasible way remains the increase of inductance l but with the stipulation that r would have to remain fixed. At Heaviside's times this was well-nigh impossible; indeed, it had to wait for the development of the new magnetic materials like *permalloy* before an effective way of "loading" the transmission line could be found.† Pupin‡ proposed the first successful installation of periodically spaced lumped *loading coils* which since that time have been used extensively on long cables and lines.§ A more effective method of loading for submarine cables is the uniform magnetic tape wrapped on the individual conductors to increase the inductance parameter.|| It should be obvious that the coil loading introduces actually a periodic line structure of lumped fourpoles alternating with transmission-line sections; this

* *Op. cit.*, Vol. I, pp. 53-77, 417-427.

† See the early memorandum by G. A. Campbell in his *Collected Works*, pp. 9-16, American Telephone and Telegraph Company, New York, 1937.

‡ M. I. Pupin, "Wave Transmission over Non-uniform Cables and Long Distance Air Lines," *Trans. AIEE*, **17**, 445-507 (1900).

§ O. E. Buckley, "The Loaded Submarine Telegraph Cable," *Bell System Tech. J.*, **4**, 355-374 (1925); T. Shaw and W. Fondillier, "Development and Application of Loading for Telephone Circuits," *Trans. AIEE*, **45**, 268-292 (1926).

|| C. E. Krarup, "Submarine Cables with Increased Inductance" [German], *Elektrotech. Z.*, **23**, 344-346 (1902).

TABLE 6.1
COMPARATIVE DATA ON TRANSMISSION LINES

Quantity	Aerial Power Line	Aerial Commun. Line		Commun. Cable		Units
		Unloaded	Ideally Loaded	Unloaded	Loaded	
Resistance r	0.538	10.15	10.15	42.1	45.7	Ω /mile
Inductance l	0.00386	0.00366	0.2928	0.001	0.039	henrys/mile
Capacitance c	0.008	0.00837	0.00837	0.062	0.062	μ f/mile
Leakance g	zero	0.29	0.29	1.5	1.5	μ mho/mile
Velocity of propa- gation						
$u = 1/\sqrt{lc}$	180,000	180,000	20,100	129,000	20,300	miles/sec
$\delta' = r/2l$	70	1386	17.4	21,050	587	sec ⁻¹
$\delta'' = g/2c$	0	17.4	17.4	12.1	12.1	sec ⁻¹
Distortion constant						
$\sigma = \delta' - \delta''$	70	1368.6	0	21,038	574.9	sec ⁻¹
Attenuation constant						
$\delta = \delta' + \delta''$	70	1403.4	17.4	21,062	599.1	sec ⁻¹
Assumed length of line s_0	100	100	100	100	100	miles
$(\delta s_0)/u$	0.0039	0.78	0.0865	16.33	2.95	
$\Delta_0 = e^{-\delta s_0/u}$	0.9961	0.458	0.917	3×10^{-4}	0.052	
Surge resistance $R_s = \sqrt{l/c}$	696	652	5830	127	794	ohms
Inductance ratio for ideal load- ing $l_L/l =$ δ'/δ''	very large	80	1	1740	48.5	
Wire gauge	0000	104 mils	104 mils	16	16	
Spacing of two wires	6 feet	12 inches	12 inches			

makes the mathematical treatment as a distortionless smooth line rather inaccurate.

To illustrate the actual conditions as found in practice, Table 6.1 gives comparative data on several typical lines as can be found in the *Electrical Engineers' Handbook*.^{*} The first column of line data is

^{*} H. Pender and W. A. Del Mar, *Electrical Engineers' Handbook, Electric Power*, 4th edition, section 14, Wiley, New York, 1949; H. Pender and K. Mellwain, *Electrical Engineers' Handbook, Electric Communication and Electronics*, 4th edition, section 10, Wiley, 1950.

for a power line designed to carry 60,000 volts. In order to be efficient for power transmission the losses must be kept low, and the attenuation factor for a length of 100 miles actually is only 0.9961. The leakance is very low, generally assumed zero, so that $\sigma = \delta = r/(2l)$. For practical purposes, one can take this line as a lossless line since in 100 miles current and voltage would decrease by less than $\frac{1}{2}\%$ in magnitude.

The second and third columns of line data pertain to an open-air communication line, unloaded and ideally loaded. The latter condition is defined as choosing an ideal inductance large enough to reduce σ to zero without increasing the losses in the system. We see that inductance and capacitance of the unloaded line have almost the same values as for the power line; however, resistance and leakance are appreciably higher. This leads to large attenuation as well as large distortion, giving almost $\delta \approx \sigma$. In a length of 100 miles, the current and voltage amplitudes would decrease to less than half values. To make this an ideally distortionless line, the inductance needs to be increased by a factor of 80. If this is done in an ideal manner, then the line presents a much reduced attenuation as seen in the fact that over a length of 100 miles the current and voltage wave amplitudes decrease only to about 92% of their initial values. However, the loading has slowed down the wave propagation markedly and has increased the surge resistance of the line, causing problems of matching.

The last two columns of line data show a communication cable with no loading and the same one with a practical type of loading, increasing the inductance only by a factor of 39 instead of 1740 as would be required for ideal loading. The unloaded cable clearly demonstrates the need for attenuation correction, since in 100 miles the amplitudes of current and voltage waves have dwindled to insignificant values. Although the actual correction falls far short of the ideal requirement, it reduces the attenuation and distortion constants by the factor 39 and thus renders this cable better than the aerial line in the values of these constants; however, the losses are still quite high and reduce current and voltage wave amplitudes to 5% over a length of 100 miles.

For practical purposes, all transmission lines appear to be best treated with the assumption $g/(2c) \ll r/(2l)$ so that one might take $g = 0$, and $\delta = \sigma$, surely not distortionless! Yet the power line is close enough to ideal lossless conditions to be approximated to a first order as a lossless line, with possible subsequent correction for the slight losses. The loaded communication lines might be approximated at first as distortionless and with considerable attenuation, with

possible subsequent correction for $\sigma \neq 0$. We shall refer back to these statements in several places where it will be necessary to decide upon practical approximations.

6.5 Traveling Waves on Finite Lossless Lines

Let us consider the finite line as shown in Fig. 6.7 with terminal impedances Z_0 and Z_t at the sending and receiving ends, respectively. For simplification we will assume the initially de-energized state so that

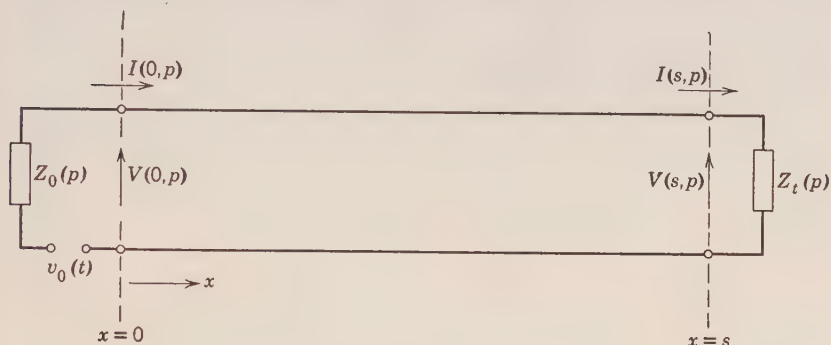


Fig. 6.7. Finite transmission line with terminal impedances at both ends.

the simpler general solutions (27) and (29) apply. The boundary conditions at the terminals are now

$$\begin{aligned} \text{at } x = 0: \quad V(0,p) &= \mathcal{L}v_0(t) - Z_0 I(0,p) \\ \text{at } x = s: \quad V(s,p) &= Z_t I(s,p) \end{aligned} \quad (46)$$

where we have to introduce from (27) and (28)

$$\begin{aligned} V(0,p) &= A + B, & V(s,p) &= Ae^{ns} + Be^{-ns} \\ I(0,p) &= \frac{1}{Z_c} (-A + B), & I(s,p) &= \frac{1}{Z_c} (-Ae^{ns} + Be^{-ns}) \end{aligned}$$

Thus we have again two equations (46) to determine the integration constants A and B . To simplify the final forms, it is desirable to define reflection coefficients, e.g.

$$\rho_0 = \frac{Z_c - Z_0}{Z_c + Z_0} \quad \rho_t = \frac{Z_c - Z_t}{Z_c + Z_t} \quad (47)$$

which, respectively, might be called sending end and receiving end *current reflection coefficients*, because $\rho_t = +1$ if $Z_t = 0$, for the short-circuited case. We obtain *voltage reflection coefficients* if we inter-

change the characteristic line and terminal impedances; we recognize that this will give the negative values of the coefficients (47).

Solving (46) for A and B and observing the definitions (47), we can write the final expressions for the voltage and current transforms

$$V(x, p) = \frac{Z_c}{Z_c + Z_0} \frac{e^{-nx} - \rho_t e^{-n(2s-x)}}{1 - \rho_0 \rho_t e^{-n2s}} \mathcal{L}v_0(t) \quad (48)$$

$$I(x, p) = \frac{1}{Z_c + Z_0} \frac{e^{-nx} + \rho_t e^{-n(2s-x)}}{1 - \rho_0 \rho_t e^{-n2s}} \mathcal{L}v_0(t) \quad (49)$$

which are the most general expressions and as such not readily amenable to quantitative evaluation.

Let us now return to the special cases we intend to treat in this section, namely the distortionless lines. For the simplest case of no losses, we can specify from (33) and (36)

$$n_0 = \frac{p}{u}, \quad \text{Re } (n_0) = \frac{1}{u} \text{Re } (p) > 0 \quad (50)$$

Because (a) the real part of n_0 will always be positive, and (b) the absolute values of the reflection coefficients must be

$$|\rho_0| < 1, \quad |\rho_t| < 1$$

since the real parts of the impedances *must* have positive values in linear passive networks, we can deduce for the denominator in (48), call it $(1 - w)$

$$|\rho_0 \rho_t e^{-n2s}| \equiv |w| < 1$$

This permits the expansion of the denominator into the binomial series

$$(1 - w)^{-1} = 1 + w + w^2 + w^3 + \dots$$

If we order the terms as multiplied by the numerator in (48) in accordance with the increasing values of the exponents, we arrive at

$$V(x, p) = \frac{Z_c}{Z_c + Z_0} \mathcal{L}v_0(t) [e^{-px/u} - \rho_t e^{-p(2s-x)/u} + \rho_0 \rho_t e^{-p(2s+x)/u} - \rho_0 \rho_t^2 e^{-p(4s-x)/u} + \rho_0^2 \rho_t^2 e^{-p(4s+x)/u} - \dots] \quad (51)$$

for the voltage transform. Similarly we can expand (49) and have for the current transform

$$I(x, p) = \frac{1}{Z_c + Z_0} \mathcal{L}v_0(t) [++++] \quad (52)$$

where the brackets contain the identical terms as in (51) but all with positive signs. The complete time solutions can be obtained by the

sum of the inverse Laplace transforms of all the individual terms of the series. It is now quite obvious that, simple as forms (51) and (52) may look, if all the impedances and therefore reflection coefficients are functions of p , the inverse transform will very quickly become complicated. Again, in principle we have the general solutions for a host of possible problems, but in practical terms we will not have unlimited courage to utilize these.

For the lossless line, as (38) shows, the characteristic impedance reduces to the surge resistance and is positive real. Let us then assume that

$$Z_0 = Z_t = \frac{1}{2}R_s, \quad \rho_0 = \rho_t = \frac{1}{3}$$

so that all the coefficients and factors in (51) and (52) are real numbers. With the step voltage $v_0(t) = V_m I$ applied at the sending end, the first term of the voltage transform series in (51) gives

$$v_1(x, t) = \mathcal{L}^{-1} \frac{2}{3} \frac{V_m}{p} e^{-px/u} = \frac{2}{3} V_m S_{-1} \left(t - \frac{x}{u} \right) \quad (53)$$

Actually this first outgoing wave from the sending end looks exactly like (41) and (42); it is the identical wave that we would find on the infinite line; surely, if there is *real* propagation, the existence of the finitely far end of the line cannot influence the initial wave emerging from the sending end terminals of the line. Any voltage form applied at the sending end would keep its exact shape and travel out with two-thirds of the applied amplitude. Obviously, at the time $t = s/u$, this first wave just arrives at the receiving end; mathematically, we have no prohibition to continue (53), but physically that part of the wave extending beyond $x = s$ has no meaning for us. All we are really interested in is the fact that the line is now covered uniformly with a voltage of value $\frac{2}{3}V_m$ and that is all the first wave conveys. Had we applied a pulse of short duration, this first wave would have carried the entire pulse beyond the line in the time $(s/u + \tau)$ where τ is the pulse duration. We also observe that the same holds for the current except that the voltage amplitude need be divided by the real characteristic impedance, exactly as for the infinite line.

The second term in (51), because of $\rho_t = \frac{1}{2}$, has the form

$$v_2(x, t) = -\mathcal{L}^{-1} \frac{1}{3} \frac{V_m}{p} e^{-p(2s-x)/u} = -\frac{1}{3} V_m S_{-1} \left(t - \frac{2s-x}{u} \right) \quad (54)$$

In form, this looks much like (53). However, the travel time $(2s-x)/u$ now *decreases with increasing x* ! Since time is irreversible in physical phenomena, we *must* interpret the form (54) consistent with

increasing time and this means we must interpret (54) as a traveling wave from the receiving end towards the sending end, starting at the receiving end $x = s$ at exactly $t = s/u$, the instant the initial wave $v_1(x, t)$ arrives! Thus $v_2(x, t)$ is the *first reflected* voltage wave, which now takes over for that part of the first wave that traveled out beyond our interest. The sign of v_2 is negative; it will therefore, as it travels back, erase just one-half of the constant voltage $\frac{2}{3}V_m$ which the first wave had laid down. The distance x decreases until it becomes zero at the sending end. This occurs at elapsed time $t = (2s)/u$, exactly twice the single travel time along the line as it should be. The same thing now happens to this wave as we let x take negative values which is possible mathematically: the wave travels beyond the line and that part has no interest for us. We now find as a combination of $v_1(x, t)$ and $v_2(x, t)$ for $t \geq (2s)/u$ a direct voltage of magnitude $\frac{1}{3}V_m$ along the line. The current wave $i_2(x, t)$ can be discussed in quite a similar manner with one reservation: its sign is positive like $i_1(x, t)$ so that the combination of these two waves results in the magnitude

$$\frac{1}{R_s} \left(\frac{2}{3} + \frac{1}{3} \right) V_m = \frac{V_m}{R_s}$$

The fact that the second and all higher terms in (51) and (52) have no values for $t < s/u$ —indeed, that each successive term takes on finite values only at time s/u after the preceding term has taken on finite values—means that for each interval $(ms)/u < t < [(m+1)s]/u$, where m is any integer, the sum of the first m terms is an *exact and complete solution* of the time-space problem! The form of the solution is called representation in terms of *traveling waves** and this approach is certainly most desirable for the recognition of the temporal sequence of phenomena on the line, at any point along the line. We can place ourselves, say at the receiving end of the line, and count the amplitudes of the successive waves, indicating the temporal sequence by arrows of travel, as e.g.

$$v(s, t) = \frac{2}{3}V_m \left(\underset{\rightarrow}{1} - \underset{\leftarrow}{\frac{1}{3}} + \underset{\rightarrow}{\frac{1}{9}} - \underset{\leftarrow}{\frac{1}{27}} + \underset{\rightarrow}{\frac{1}{81}} - \cdots \right)$$

and compute the final steady-state value as

$$\begin{aligned} v(s, \infty) &= \frac{2}{3}V_m \left[\left(1 - \frac{1}{3} \right) \left(1 + \frac{1}{9} + \frac{1}{81} + \cdots \right) \right] \\ &= \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{9}{8} V_m = \frac{1}{2} V_m \end{aligned}$$

* Heaviside, *op. cit.*, Vol. II, pp. 69–76.

For the current we have

$$i(s, \infty) = \frac{2}{3} \frac{V_m}{R_s} \left(1 + \frac{1}{3} + \frac{1}{9} + \cdots \right) = \frac{V_m}{R_s}$$

This checks the normal steady-state computation for the final voltage and current distribution. Of course, for larger values of time, the series (51) and (52) have little practical value unless the series converge rather rapidly.

Perhaps it is important here to come back to some of the discussion in section 6.1. We have assumed here, as also indicated in Fig. 6.7, well-defined planes $x = 0$ and $x = s$ as the junctions of the "transmission-line section" $0 < x < s$ and the "terminal sections" $x < 0$ and $x > s$ where we lumped the entire electromagnetic phenomena neatly into impedance values Z_0 and Z_t . It must be obvious that that is a crude approximation at best and we can hope to find satisfactory agreement between theory and experiment only for relatively long lines or very low frequencies. Much of the confirmation of traveling waves on almost lossless lines comes, of course, from power lines where we would expect good agreement.*

Important other special values of the terminal resistances are $Z_t = 0$, the short-circuited line, or $Z_t = \infty$, the open-circuited line. Assume $Z_0 = \alpha R_s$ where $\alpha \ll 1$ might be a small number to account for the actual small losses of the line and take $Z_t = 0$, the *short-circuited* case for which $\rho_t = 1$. The voltage transform (51) can now be considered in wave pairs, each of these canceling the other to maintain the required zero voltage at the short-circuited receiving end. At the sending end, the first term will be maintained, and the successive pairs beyond this first term cancel in pairs upon reflection at $x = 0$. On the other hand, the current will increase continuously, since all the terms are positive. We find as the steady-state short-circuit current

$$i(0, \infty) = i(s, \infty) = \frac{V_m}{(1 + \alpha)R_s} [2 + 2\rho_0 + 2\rho_0^2 + \cdots] = \frac{V_m}{\alpha R_s}$$

tending to infinite values, if the series resistance goes to zero as one would expect.

For the *open-circuited* case we have $\rho_t = -1$, so that the voltage terms change sign in pairs. Here, the voltage at the receiving end builds up upon reflection to twice its normal value, indicating the

* V. Bush, "Transmission Line Transients," *Trans. AIEE*, **42**, 878-901 (1923); Bewley,^{c1} chapters 3 and 4; C. E. Magnusson, *Electric Transients*, chapter VI, McGraw-Hill, New York, 1926.

known danger of transient overvoltages on power lines if discontinuities of high resistance values arise. With $Z_0 = \alpha R_s$ as before, the final receiving end voltage is the same as the applied voltage V_m because no steady-state current is flowing on the line.

6.6 Traveling Waves on Finite Distortionless Lines

As shown for the infinite distortionless line, the propagation function n_d has a real positive part, in fact larger than the lossless line, so that we can employ the same expansions (51) and (52) for the finite distortionless lines with losses. To take into account the attenuation, we need only replace p in the brackets of (5) by $(p + \delta)$ and otherwise accept the results from section 6.5.

Thus, let us apply a step voltage $V_m l$ at the sending end of a finite distortionless line as in Fig. 6.7. We take the characteristic impedance from (38) again as R_s , real and positive, and assume $Z_0 = Z_t = \frac{1}{2}R_s$, so that we have $\rho_0 = \rho_t = \frac{1}{3}$, also as before. The first term of (51) will now read

$$V_1(x, p) = \frac{2}{3} \frac{V_m}{p} e^{-\frac{\delta x}{u}} e^{-p \frac{x}{u}}$$

with the inverse transform of the same form as in (53) except for the additional factor $e^{-\delta(x/u)}$ which defines the attenuation along the line. In turn, this result is merely a special case of the general form (45) for the infinite line, namely

$$v_1(x, t) = \frac{2V_m}{3} e^{-\frac{\delta x}{u}} S_{-1} \left(t - \frac{x}{u} \right) \quad (55)$$

As indicated in (45), any voltage form whatsoever would keep its precise shape as time function but would be attenuated as it traveled along the line. When $t = s/u$, this wave (55) will travel on beyond our range of interest, but it has laid down along the line a voltage constant in time and decreasing exponentially from the sending end at the rate $e^{-\delta(x/u)}$. We shall designate the total attenuation for the length of the line $e^{-\delta(s/u)} = \Delta$.

The second term in (51), because $\rho_t = \frac{1}{3}$, has the inverse transform

$$v_2(x, t) = -\frac{2V_m}{9} e^{-\delta \frac{2s-x}{u}} S_{-1} \left(t - \frac{2s-x}{u} \right) \quad (56)$$

This signifies, as outlined in connection with (54), a *reflected wave* starting at $t = s/u$ and traveling from the receiving end towards the

sending end, but again being attenuated at the same rate as the first outgoing wave. This means that this returning wave does *not* uniformly erase part of the first wave, but rather leaves more and more of the original wave standing as it approaches the sending end. It reaches the sending end at $t = (2s)/u$ with a magnitude

$$v_2\left(0, \frac{2s}{u}\right) = -\frac{2V_m}{9}\Delta^2$$

so that the total voltage along the line at time $t = (2s)/u$ will have the distribution

$$v_1\left(x, \frac{2s}{u}\right) + v_2\left(x, \frac{2s}{u}\right) = V_m\left[\frac{2}{3}e^{-\frac{\delta x}{u}} - \frac{2}{9}\Delta^2e^{\frac{\delta x}{u}}\right] \tag{57}$$

This is essentially the same form that we expect as the final steady-state distribution in terms of “outgoing and reflected” waves, but taken in a different sense! The solutions (55) and (56) are the *dynamic* elements of the voltage build-up process, the actual traveling waves; whereas (57) is the sum of their static values, once the elapsed time has increased beyond the respective travel time along the line.

The following third term of (51) reads with the added attenuation factor

$$v_3(x,p) = \frac{2}{3}\frac{V_m}{p}\frac{1}{3}\frac{1}{3}e^{-\frac{\delta(2s+x)}{u}}e^{-p\frac{2s+x}{u}} = \frac{2}{27}\frac{V_m}{p}\Delta^2e^{-\frac{\delta x}{u}}e^{-p\frac{2s+x}{u}}$$

which clearly shows the cumulative effect of the attenuation with total travel distance of the successive waves. In order to see the history of these waves and the decrease of their amplitudes better, we might evolve a tabular schematic which portrays the traveling wave expansions (51) and (52) for the two end points of the line $x = 0$ and $x = s$. At these points, the series terms in the brackets are valid for all time from the instant of their inception. This time instant in turn is an integral multiple of $T = s/u$, the time for one travel along the line. Table 6.2 actually gives this schematic for voltage and current waves. It can be interpreted as giving the individual wave factors in Laplace transform notation to the uniformly applying factors outside the bracket; or it can be interpreted as giving directly the factors for the source voltage time function, if we are dealing only with resistive terminal impedances. For lossless lines $\Delta = 1$ throughout; for distortionless but attenuating lines $\Delta < 1$ is given with the line data.

TABLE 6.2
SCHEMATIC TABULATION OF TRAVELING WAVES. EXPANSIONS (51) AND (52)
FOR CONTINUOUSLY ACTIVE SOURCE VOLTAGE

Time t/T where $T = s/u$	Series Terms for Distortionless Line at	
	Sending End $x = 0$	Receiving End $x = s$
1	$+1 \rightarrow$	
2		$\rightarrow +\Delta$ $\leftarrow \mp \rho_t \Delta$
3	$\mp \rho_t \Delta^2 \leftarrow$ $+ \rho_0 \rho_t \Delta^2 \rightarrow$	
4		$\rightarrow + \rho_0 \rho_t \Delta^3$ $\leftarrow \mp \rho_0 \rho_t^2 \Delta^3$
5	$\mp \rho_0 \rho_t^2 \Delta^4 \leftarrow$ $+ \rho_0^2 \rho_t^2 \Delta^4 \rightarrow$	

Note: Where double signs are shown, the upper sign holds for the voltage, the lower for the current.

In the example just treated, we had only resistive terminations with a step voltage applied at the sending end. The total voltage at the sending end is now the sum of all the terms in the respective column of Table 6.2, each term delayed by the time indicated on the left. Thus

$$v(0, t) = \frac{2}{3} V_m [S_{-1}(t) - (\frac{1}{3} - \frac{1}{9}) \Delta^2 S_{-1}(t - 2T) - \frac{1}{9} (\frac{1}{3} - \frac{1}{9}) \Delta^4 S_{-1}(t - 4T) - \dots]$$

whereas for the current we have all positive terms, namely

$$i(0, t) = \frac{2}{3} \frac{V_m}{R_s} \left[S_{-1}(t) + \left(\frac{1}{3} + \frac{1}{9} \right) \Delta^2 S_{-1}(t - 2T) + \frac{1}{9} \left(\frac{1}{3} + \frac{1}{9} \right) \Delta^4 S_{-1}(t - 4T) + \dots \right]$$

As the current continuously increases, the voltage at the line terminals must decrease because of the increasing voltage drop across the terminal resistance $R_0 = \frac{1}{2} R_s$. The ultimate values can be found by the series

$$v(0, \infty) = \frac{2}{3} V_m \left(1 - \frac{2}{9} \Delta^2 - \frac{2}{9^2} \Delta^4 - \frac{2}{9^3} \Delta^6 - \dots \right) = \frac{2}{3} V_m \frac{1 - (\Delta^2/3)}{1 - (\Delta^2/9)}$$

If we assign the values $\Delta^2 = 1, 0.81$, and 0.09 , we obtain respectively for the values of the last fraction $0.75, 0.803$, and 0.982 . For the current value we obtain similarly

$$i(0, \infty) = \frac{2}{3} V_m \left(1 + \frac{4}{9} \Delta^2 + \frac{4}{9^2} \Delta^4 + \frac{4}{9^3} \Delta^6 + \dots \right)$$
$$= \frac{2}{3} V_m \frac{1 + (\Delta^2/3)}{1 - (\Delta^2/9)}$$

with the respective values for the last fraction 1.50, 1.395, 1.04.

An important special case might be the *open-circuited* line $Z_t = \infty$, with $Z_0 = 0$, the step voltage directly applied at the terminals of the

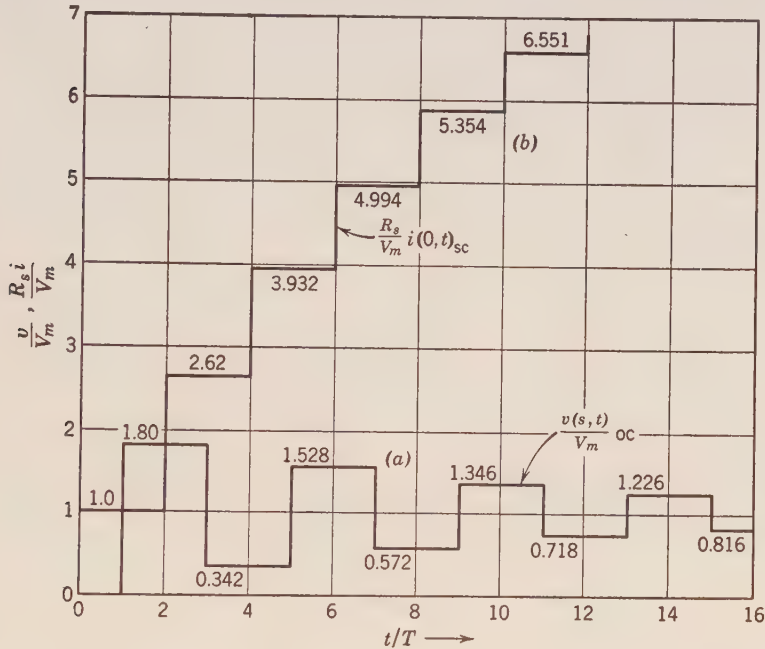


Fig. 6.8. Response to a step surge of a finite distortionless line with $\Delta = 0.9$. (a) Receiving end voltage for open-circuited line; (b) sending end current for short-circuited line.

distortionless line. We have

$$\frac{Z_c}{Z_c + Z_0} = 1, \quad \rho_0 = 1, \quad \rho_t = -1$$

so that for the receiving end voltage we read from (51) and Table 6.2 the complete solution

$$v(s, t) = V_m [(\Delta + \Delta) S_{-1}(t - T) - (\Delta^3 + \Delta^3) S_{-1}(t - 3T) + \dots]$$

Fig. 6.8 gives the oscillation of this voltage value for $\Delta = 0.9$; for actual power lines Δ can be close to unity and easily produce almost

twice the normal voltage at the end of the open-circuited line. The final steady-state value is easily found

$$v(s, \infty) = 2V_m(\Delta - \Delta^3 + \Delta^5 - \Delta^7 + \cdots) = \frac{2\Delta}{1 + \Delta^2} V_m$$

For $\Delta = 0.9$ chosen for the figure, we obtain $0.995V_m$.

The entering current for the short-circuited line is also of particular interest. With $Z_t = 0$ and $Z_0 = 0$, we have here

$$\rho_0 = +1, \quad \rho_t = +1$$

so that

$$i(0, t) = \frac{V_m}{R_s} [S_{-1}(t) + 2\Delta^2 S_{-1}(t - 2T) + 2\Delta^4 S_{-1}(t - 4T) + \cdots]$$

which is also shown in Fig. 6.8 for the same value $\Delta = 0.9$ as the open-circuit voltage. The final value of this short-circuit current is

$$i(0, \infty) = \frac{V_m}{R_s} (1 + 2\Delta^2 + 2\Delta^4 + \cdots) = \frac{V_m}{R_s} \cdot \frac{1 + \Delta^2}{1 - \Delta^2}$$

For $\Delta = 0.9$ this gives the large value 9.53. If we recall that forces increase with the square of the current values, we appreciate the many mechanical problems that attend the sudden short circuit on transmission lines.

Since any distortionless line has as characteristic impedance its surge resistance R_s , the direct junction of two such lines can be treated as a resistive termination of the first line. For the propagation of the current and voltage waves on the second line we need only treat the output waves of the first line as new input waves.

As a further example, let us apply a rectangular pulse of duration τ to the same distortionless line as shown in Fig. 6.7 terminated at both ends into resistances of half the value of the surge resistance, so that again

$$Z_0 = Z_t = \frac{1}{2}R_s, \quad \rho_0 = \rho_t = \frac{1}{3}$$

The input pulse is given by the superposition of two step voltages

$$V_0(t) = V_m[S_{-1}(t) - S_{-1}(t - \tau)]$$

each of which can be identified (except for the sign and the delay) with the illustrative example given at the beginning of this section. The first outgoing wave will therefore be in accordance with (55)

$$v_1(x, t) = \frac{2}{3} V_m e^{-\delta \frac{x}{u}} \left[S_{-1} \left(t - \frac{x}{u} \right) - S_{-1} \left(t - \tau - \frac{x}{u} \right) \right] \quad (58)$$

If the pulse duration τ is smaller than the travel time $T = s/u$ once down the line, then oscillograms along the line will show this distinct first pulse with exponentially decreasing amplitude and with delay defined by the local travel time. However, as we approach the receiving end of the line, the reflected pulse given by (56) with the appropriate adaptation

$$v_2(x,t) = -\frac{1}{3} V_m e^{-\delta \frac{2s-x}{u}} \left[S_{-1} \left(t - \frac{2s-x}{u} \right) - S_{-1} \left(t - \tau - \frac{2s-x}{u} \right) \right] \tag{59}$$

will make its appearance before the first pulse is finished, so that an apparent distortion takes place. This effect will, of course, be a function of the pulse duration.

To see this better, let us choose a definite point of observation,* say $x = s/3$. The successive single traveling voltage step waves have amplitudes defined by the expansion (51) if we add the appropriate attenuation factors. With the definition $e^{-\delta(s/u)} = \Delta$ we can write down the table of successive amplitudes for $\delta(s/u) = 0.12$ in Table 6.3.

TABLE 6.3
AMPLITUDES OF TRAVELING STEP WAVES FROM (51) AT $x = s/3$

Time of Arrival	Outgoing direction	Amplitude	Numerical Value	Return Direction
$\frac{1}{3}T$	→	$+\frac{2}{3}\Delta^{\frac{1}{3}}$	$= +0.6405$	
$\frac{2}{3}T$		$-\frac{2}{3} \cdot \frac{1}{3}\Delta^{\frac{5}{3}}$	$= -0.1819$	←
$\frac{7}{3}T$	→	$+\frac{2}{3}(\frac{1}{3})^2\Delta^{\frac{7}{3}}$	$= +0.0560$	
$\frac{11}{3}T$		$-\frac{2}{3}(\frac{1}{3})^3\Delta^{1\frac{1}{3}}$	$= -0.0159$	←
$\frac{13}{3}T$	→	$+\frac{2}{3}(\frac{1}{3})^4\Delta^{1\frac{2}{3}}$	$= +0.0049$	

For a pulse duration $\tau = T/2$ we find clear separation of the individual waves as shown in Fig. 6.9. The time distance between the individual pulses can actually be taken as a measure of the location of the point of observation. Conversely, one can take oscillograms of pulses generated by faults along a transmission line and determine the location of the fault.† For a pulse duration $\tau = 3T$ we find quite a different picture as shown in Fig. 6.10. Here the reflections all merge into one signal which would be interpreted as a distorted

* E. Weber, "Traveling Waves on Transmission Lines," *Elec. Eng.*, **61**, 302-309 (1942).

† L. G. Abraham, A. W. Lebert, J. B. Maggio, and J. T. Schott, "Pulse Echo Measurements on Telephone and Television Facilities," *Trans. AIEE*, **66**, 541-548 (1947). R. F. Stevens and T. W. Stringfield, "A Transmission Line Fault Locator Using Fault Generated Surges," *Ibid.*, **67**, 1168-1178 (1948).

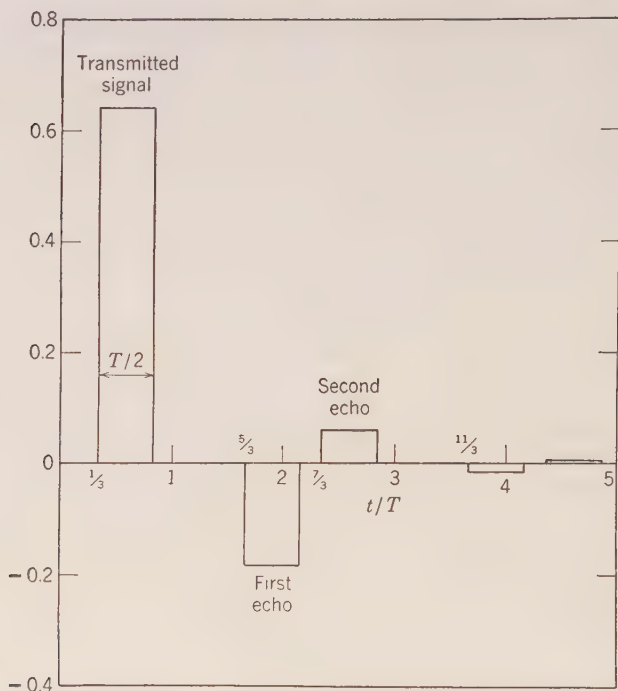


Fig. 6.9. Pulse reflections on finite distortionless transmission line if pulse duration $\tau = T/2$. (Reproduced with permission from *Electrical Engineering*, June 1942.)

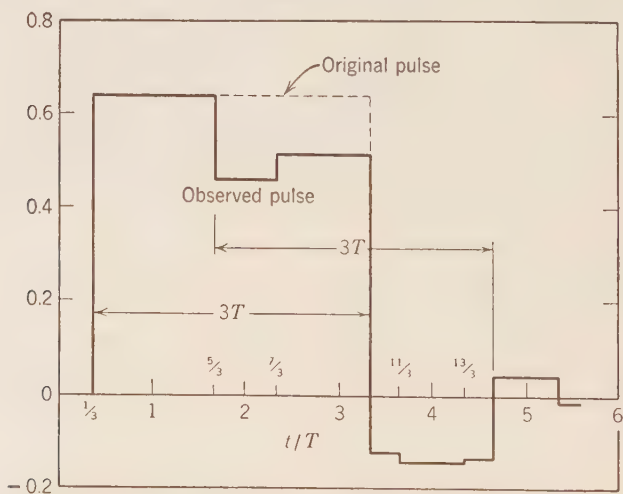


Fig. 6.10. Pulse distortion by reflection on finite distortionless transmission line for pulse duration $\tau = 3T$. (Reproduced with permission from *Electrical Engineering*, June 1942.)

reproduction of the original pulse with a tail of irregular shape. For a train of repetitive pulses it is important to choose the individual pulse duration as well as the pulse spacing so as to avoid interference of the echos with the succeeding originals which obviously could result in rather severe distortion and consequent unintelligibility.

The original transmission-line equations (21) were based upon a two-conductor system. If more than two conductors are involved, as, e.g., in a three-phase three- or four-wire system, or in a system of parallel telephone wires, then relations (21) must be expanded to take into account the mutual electrostatic as well as magnetic couplings of the lines. In this case, several modes of propagation will exist quite analogous to the transient modes of a network composed of several meshes. This will result in so-called *multivelocity* waves which necessitate a more careful examination of the oscillograms for fault location.* A rather extensive treatment of multivelocity traveling waves is given in Bewley,^{C1} chapter 6.

6.7 Sinusoidal Traveling Waves

So far we have studied only waves of constant magnitude. It is of interest and importance to analyze the building-up of the response to time-variable source functions. As a simple illustration, let us consider a conventional sine wave source voltage

$$v_0(t) = V_m \sin(\omega t + \psi) \quad (60)$$

To obtain the response of the line, we can still use the traveling wave expansions (51) and (52) which are particularly appropriate to find the initial build-up of current and voltage anywhere along the line. The first outgoing voltage wave has the Laplace transform from (51)

$$V_1(x, p) = \frac{Z_c}{Z_c + Z_0} e^{-\delta \frac{x}{u}} e^{-p \frac{x}{u}} \mathcal{L} v_0(t) \quad (61)$$

where we inserted the damping factor $e^{-\delta \frac{x}{u}}$ as required for the distortionless line. If we stipulate here $Z_0 = 0$, $Z_t = \frac{1}{4}R_s$, then $\rho_0 = 1$, $\rho_t = \frac{3}{5}$ from (47), and the impedance ratio becomes unity. Thus, as we see from Table 1.4, the inverse Laplace transform of (61) is simply the delayed and reduced reproduction of the identical sine wave (60), namely

$$v_1(x, t) = e^{-\delta \frac{x}{u}} V_m \sin \left[\omega \left(t - \frac{x}{u} \right) + \psi \right] S_{-1} \left(t - \frac{x}{u} \right) \quad (62)$$

* L. J. Lewis, "Traveling Wave Relations Applicable to Power-System Fault Locators," *Ibid.*, **70**, 1671-1678 (1951).

If we take an oscillogram at the receiving end $x = s$, we observe absolute zero during the time interval $T = s/u$ after the voltage has been applied at the sending end and then a sudden rise to $\Delta V_m \sin \psi$ where $\Delta = e^{-\delta(s/u)}$, the total attenuation along the line. But this received first wave remains a local time function

$$v_1(s, t') = \Delta V_m \sin (\omega t' + \psi) \quad t' > 0 \quad (63)$$

where $t' = (t - T)$ is the local time counted from the instant of the arrival of the first voltage wave. At the same instant $t = T$ or $t' = 0$, the "reflected" voltage wave starts in, which is the inverse transform of the second terms in (51) with the added attenuation factor, namely

$$v_2(x, t) = -\frac{3}{5} e^{-\delta \frac{2s-x}{u}} V_m \sin \left[\omega \left(t - \frac{2s-x}{u} \right) + \psi \right] S_{-1} \left(t - \frac{2s-x}{u} \right)$$

Locally at $x = s$ this becomes, expressed again in time $t' = t - T$

$$v_2(s, t') = -\frac{3}{5} \Delta V_m \sin (\omega t' + \psi) \quad t' > 0$$

This wave combines with (63) to give the total exact voltage at the receiving end for the time interval $0 < t' < 2T$

$$v_{1,2}(s, t') = +\frac{2}{5} \Delta V_m \sin (\omega t' + \psi) \quad t' > 0 \quad (64)$$

Actually, this voltage will remain indefinitely at the receiving end; however, at time $t' = 2T$ the third wave in (51) arrives and at the same instant its reflection starts in, given by the fourth term. As before, we can combine the inverse Laplace transforms of these two terms at $x = s$ into

$$v_{3,4}(s, t') = +\frac{6}{25} \Delta^3 V_m \sin [\omega(t' - 2T) + \psi] \quad t' > 2T \quad (65)$$

The total and exact voltage at the receiving end is now for $0 < t' < 4T$ the sum of (64) and (65) as shown in Fig. 6.11. For this graphical demonstration we abbreviated $\frac{2}{5} \Delta V_m = A$ as the arbitrary amplitude reference factor and used the numerical values: $\omega T = \pi/6$, $\psi = \pi/6$, and $(\delta s)/u = 0.12$, $\Delta^2 = 0.787$. The process has been continued in Table 6.4 and, at every interval $2T$, another, smaller term pair adds to the sum total of all foregoing. For comparison, Fig. 6.11 also shows the original voltage wave form applied at the sending end but with amplitude A rather than $V_m = (5/2\Delta)A = 2.82A$ in order to keep it within the scale of the graph.

The final wave form at the receiving end must, of course, be a pure sine wave whose amplitude can be found from ordinary a-c steady-state analysis. We need only return to (48), replace the voltage trans-

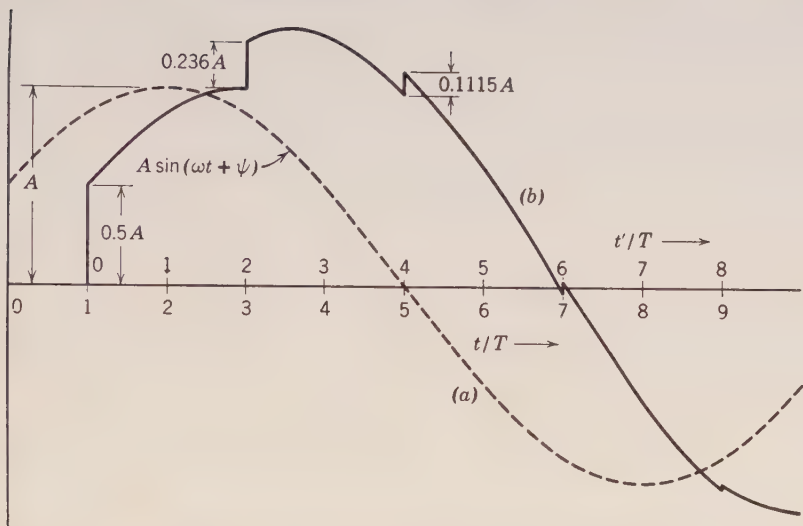


Fig. 6.11. Building-up of sinusoidal response on finite distortionless transmission line. (a) Wave shape of voltage at sending end; (b) at receiving end.

TABLE 6.4
BUILDING-UP OF SINUSOIDAL VOLTAGE AT RECEIVING END SHOWN IN FIG. 6.11

Order Number of Term From (51)	Amplitude	Phase Delay With Respect to Sending End Voltage	Start of Wave at	Absolute Phase angle
1 + 2	$\frac{2}{5} \Delta V_m = A$	$-\omega T = -\frac{\pi}{6}$	$t = T$	$\psi - \omega T = 0$
3 + 4	$\left(\frac{3}{5} \Delta^2\right) A = 0.472A$	$-3\omega T = -\frac{\pi}{2}$	$t = 3T$	$\psi - 3\omega T = -\frac{\pi}{3}$
5 + 6	$\left(\frac{3}{5} \Delta^2\right)^2 A = 0.223A$	$-5\omega T = -\frac{5\pi}{6}$	$t = 5T$	$\psi - 5\omega T = -\frac{2\pi}{3}$
7 + 8	$\left(\frac{3}{5} \Delta^2\right)^3 A = 0.105A$	$-7\omega T = -\frac{7\pi}{6}$	$t = 7T$	$\psi - 7\omega T = -\pi$
9 + 10	$\left(\frac{3}{5} \Delta^2\right)^4 A = 0.0496A$	$-9\omega T = -\frac{3\pi}{2}$	$t = 9T$	$\psi - 9\omega T = -\frac{4\pi}{3}$

forms by the voltage phasors, and substitute in the transfer function $j\omega$ for p . The phasor of the applied voltage is easily obtained if we write (60)

$$v_0(t) = \text{Im} (V_m e^{j\psi} e^{j\omega t})$$

so that

$$V_0 = V_m e^{j\psi} \tag{66}$$

The phasor of the voltage anywhere along the line is defined as function of x , namely

$$v(x, t) = \text{Im} [V(x)e^{j\omega t}] \quad (67)$$

We thus obtain from (48) with $Z_0 = 0$, $\rho_0 = 1$, $\rho_t = \frac{2}{3}$ as assumed above

$$V(x) = \frac{e^{-\frac{x}{u}(j\omega + \delta)} - \frac{2}{3}e^{-\frac{2s-x}{u}(j\omega + \delta)}}{1 - \frac{2}{3}e^{-\frac{2s}{u}(j\omega + \delta)}} V_0 \quad (68)$$

For $x = s$ and introducing Δ , we have by rationalization at the receiving end of the line

$$V(s) = \frac{2}{3}\Delta[(1 - \frac{2}{3}\Delta^2)^2 \cos^2 \omega T + (1 + \frac{2}{3}\Delta^2)^2 \sin^2 \omega T]^{-1/2} e^{-j\theta} V_0$$

The phase shift of the received voltage with respect to the applied voltage is

$$\theta = \tan^{-1} \left(\frac{1 + \frac{2}{3}\Delta^2}{1 - \frac{2}{3}\Delta^2} \tan \omega T \right) = 58^\circ 9'$$

if we use the numerical values from above. Figure 6.11 verifies this phase shift, because the time difference is almost $2T$ which means an angular phase shift $2\omega T = \pi/3$.

As our example indicated, all the component waves at any fixed location x along the line are sinusoids and can therefore be represented in complex notation by means of the corresponding phasors. We could also use (68) and expand in the same manner as (51) which would give the same resolution into the component phasors. Utilizing the phasor diagram, we can construct graphically the received voltage wave at $x = s$. The base circle of radius A in Fig. 6.12 represents the path of the uniformly rotating phasor $V_{1,2}$ whose instantaneous projection on the vertical axis represents $A \sin \omega t$. The first component sinusoid from Table 6.4 has zero absolute phase angle so that its phasor $V_{1,2}$ is extended along the axis $t = 0$. Its physical values appear at $\omega t = \omega T = \pi/6$, and the projection has value $\frac{1}{2}A$ which is the initial rise of the received voltage in Fig. 6.11. As the end point of the phasor revolves from a' to a'' , its projection above the linear time scale in Fig. 6.11 describes the first sinusoidal arc. At time $t = 3T$ the second component sinusoid from Table 6.4 comes into existence. Its phasor is extended along the absolute phase direction $-\pi/3$ and is shown as $V_{3,4}$ in Fig. 6.12. The numerical addition of the sinusoidal responses which we performed for Fig. 6.11 can be achieved much simpler graphically, by adding in the phasor sense

$V_{1,2} + V_{3,4}$ leading to the end point b . Because this resultant wave starts with a delay of $3T$, we proceed along the circle from b to b' which marks the starting point of the physical existence, and again the projection of the circular arc from b' to b'' above the linear time scale in Fig. 6.11 describes the second sinusoidal arc with the vertical sudden rise of $0.236A$ at its beginning. This process can now be continued with the phasor $V_{5,6}$ as shown in Fig. 6.12.

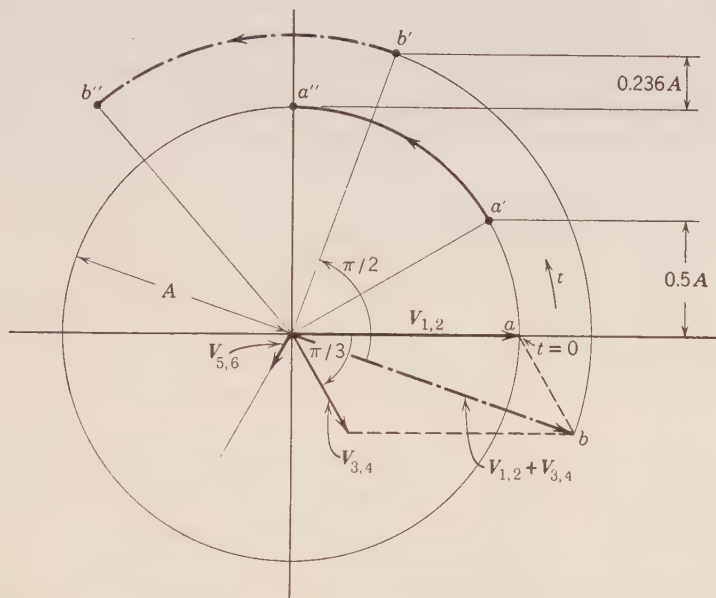


Fig. 6.12. Graphical construction of resultant voltage phasor at the receiving end for Fig. 6.11.

This graphical construction can also be used for the current response because the characteristic impedance of the distortionless line is real and thus does not affect the sine wave character of the individual traveling wave components. Comparison of the responses of open- and short-circuited distortionless lines obtained by such graphical construction with oscillograms on an artificial line simulating the long line show very good agreement.*

6.8 Distortion of Traveling Waves by Lumped Elements

The terminations of the transmission lines have been assumed to be resistive because this makes the reflection factors (47) real numbers

* E. Weber and F. E. Kulman, "Sinusoidal Traveling Waves," *Trans. AIEE*, **55**, 245-251 (1936).

and thus preserves the signal shape for every one of the terms in expansions (51) and (52). If we admit more general impedance terminations, then the signal will suffer distortion upon any single reflection even though the transmission along the lossless and distortionless lines themselves will remain distortionless.

As a simple practical example assume a finite lossless line of parameters l_1, c_1 connected through a lumped inductance L to an infinite lossless line of parameters l_2, c_2 . Fig. 6.13 shows the schematic arrangement in which inductance L has been divided equally between go and return wire. We must assume that the output terminals $x_1 = s$ are for all practical purposes so close to the input terminals $x_2 = 0$ of the second line that we can disregard the severe problem of

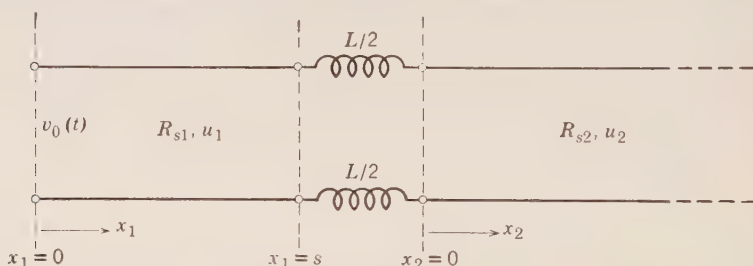


Fig. 6.13. Connection of finite and infinite transmission lines through a lumped series inductance.

field redistribution. We will not even try to justify our restriction to the principal mode propagation along both transmission lines.

The solution of the current and voltage distribution along the second infinite line is identically given by (37), or with the proper indices

$$V(x_2, p) = D e^{-p \frac{x_2}{u_2}}, \quad I(x_2, p) = \frac{D}{R_{s2}} e^{-p \frac{x_2}{u_2}} \quad (69)$$

where the constant D must be found from the continuity of voltage and current at the terminals $x_2 = 0$. We observe, however, that independent of D the ratio of voltage to current transforms is the surge resistance R_{s2} , so the second line can be represented by a series resistance termination at $x_2 = 0$, or the first line can be assumed to have the terminal impedance

$$Z_{t1} = R_{s2} + pL \quad (70)$$

With this, we can now write the solution for current and voltage along the first transmission line directly as in (48) and (49), or also as expanded in (51) and (52), where

$$\rho_0 = 1, \quad \rho_t = \frac{R_{s1} - R_{s2} - pL}{R_{s1} + R_{s2} + pL} = \frac{\beta - p}{\alpha + p} \quad (71)$$

if we introduce further

$$\alpha = \frac{R_{s1} + R_{s2}}{L}, \quad \beta = \frac{R_{s1} - R_{s2}}{L} \quad (72)$$

The more interesting case for power transmission lines is the study of propagation and deformation of switching surges. Suppose, therefore, that a step voltage is applied directly at the terminals $x_1 = 0$ of the first line, so that $v_0(t) = V_m 1$ with the Laplace transform

$$\mathfrak{L}v_0(t) = \frac{V_m}{p}$$

The expansion (51) for the voltage becomes explicitly

$$V(x_1, p) = \frac{V_m}{p} \left[e^{-p(x_1/u_1)} + \frac{p - \beta}{p + \alpha} e^{-p(2s-x_1)/u_1} - \frac{p - \beta}{p + \alpha} e^{-p(2s+x_1)/u_1} \right. \\ \left. - \left(\frac{p - \beta}{p + \alpha} \right)^2 e^{-p(4s-x_1)/u_1} + \left(\frac{p - \beta}{p + \alpha} \right)^2 e^{-p(4s+x_1)/u_1} + \dots \right] \quad (73)$$

The first term is, of course, again the same as on an infinitely long line. Its inverse Laplace transform

$$v_1(x_1, t) = V_m S_{-1} \left(t - \frac{x_1}{u_1} \right) \quad (74)$$

represents a step voltage traveling out towards the load end of the line, arriving there at $t = s/u_1 = T$. At that same instant, the second term of (73) comes into existence as the "reflected" wave, but its shape is the inverse Laplace transform

$$v_2(x_1, t) = \mathfrak{L}^{-1} \frac{V_m}{p} \frac{p - \beta}{p + \alpha} e^{-p(2s-x_1)/u_1}$$

We find the transform for the rational fraction quite readily by means of Table 1.3 if we separate into the two terms

$$\mathfrak{L}^{-1} \left(\frac{1}{p + \alpha} - \frac{\beta}{p(p + \alpha)} \right) = e^{-\alpha t} - \frac{\beta}{\alpha} (1 - e^{-\alpha t})$$

The total transform is then with the delay indicated by the exponential factor (see Table 1.4)

$$v_2(x_1, t) = \frac{V_m}{\alpha} [(\alpha + \beta)e^{-\alpha(t-\theta_2)} - \beta] S_{-1}(t - \theta_2) \quad (75)$$

where $\theta_2 = (2s - x_1)/u_1$. This reflected wave superimposes then upon the initial uniform voltage (74) along the line a time-variable voltage distribution, starting abruptly at any point x_1 with the initial value V_m at the time $t = \theta_2$, and then decaying exponentially, approaching at time $t \rightarrow \infty$ the value

$$-\frac{\beta}{\alpha} V_m = \frac{R_{s2} - R_{s1}}{R_{s2} + R_{s1}} V_m$$

The reflection has initially the same character as from an open-circuited line which is quite consistent since the series inductance permits no sudden finite current value.

The third term in (73) has the same form as the second except for the longer time delay and, of course, represents the reflection of the second wave from the generator end of the first line. Since v_2 arrives at $x_1 = 0$ with initial value $+V_m$, the reflected wave must be negative of the same value in order to maintain the original terminal voltage. We can write at once this third traveling wave by reference to (75)

$$v_3(x_1, t) = -\frac{V_m}{\alpha} [(\alpha + \beta)e^{-\alpha(t-\theta_3)} - \beta]S_{-1}(t - \theta_3) \quad (76)$$

Because of the time dependence, $v_2(x_1, t)$ and $v_3(x_1, t)$ will cancel completely only at the generator end $x_1 = 0$; everywhere else along the line, a time-varying residual will remain. However, as $t \rightarrow \infty$, $v_3(x_1, t)$ will assume the negative same value as $v_2(x_1, t)$, so that eventually only the first uniform voltage distribution remains, as we would expect for lossless transmission lines.

It is obvious now that the pairs of successive terms in (73) will produce a transient variation of the voltage distribution along the line, but, because of the lossless character, will eventually cancel completely. We recognize also that the successive terms will be more and more complex in their wave shapes but will always start with the full value of the original voltage wave. Thus, the fourth term in (73) requires the inverse Laplace transform of

$$F(p) = \frac{1}{p} \left(\frac{p - \beta}{p + \alpha} \right)^2$$

We can take the general form of residue from (1.55) for $p = 0$ but must use (1.69) with $r = 2$ for the pole of second-order $p = -\alpha$, so that

$$\mathcal{L}^{-1}F(p) = \left(\frac{\beta}{\alpha} \right)^2 + \left[\frac{d}{dp} \left(\frac{1}{p} (p - \beta)^2 e^{pt} \right) \right]_{p=-\alpha}$$

The differentiation gives, because of the triple product

$$\left(-\frac{1}{p^2} (p - \beta)^2 + \frac{2}{p} (p - \beta) + \frac{t}{p} (p - \beta)^2 \right) e^{pt}$$

Introducing $p = -\alpha$ and combining with the first term leads to

$$\mathfrak{L}^{-1}F(p) = 1 - \frac{\alpha^2 - \beta^2}{\alpha^2} (1 - e^{-\alpha t}) - \left(\frac{\alpha + \beta}{\alpha} \right)^2 \alpha t e^{-\alpha t}$$

and therefore for the component voltage wave traveling back to the generator end

$$v_4(x_1, t) = -V_m \left[1 - \frac{\alpha^2 - \beta^2}{\alpha^2} [1 - e^{-\alpha(t-\theta_4)}] - \left(\frac{\alpha + \beta}{\alpha} \right)^2 \alpha (t - \theta_4) e^{-\alpha(t-\theta_4)} \right] \quad t \geq \theta_4 \quad (77)$$

where $\theta_4 = (4s - x_1)/u_1$. At $t = \theta_4$, the starting time of this wave anywhere along the line, it has the value V_m and as $t \rightarrow \infty$ it reduces to the steady value

$$-\left(\frac{\beta}{\alpha} \right)^2 V_m = -\left(\frac{R_{s2} - R_{s1}}{R_{s2} + R_{s1}} \right)^2 V_m$$

We could, of course, take both initial and final values of all the component waves directly from expansion (73) if we make use of the general relations 1a and 1b from Table 1.4. In the first case we must disregard the exponential delay factors since they do not affect the initial values. Fig. 6.14 graphically illustrates the foregoing results by picturing the distributions of the resultant voltage wave along the line for instances corresponding to the arrival of the individual waves at the mid-point of the first line $x_1 = s/2$. For the actual computation we have assumed $\alpha = 1/(5T)$, $\beta = 1/(10T)$, with $T = s/u_1$ the single travel time along the line. Comparison shows that v_4 decreases faster than v_2 .

To evaluate the voltage distribution along the second line, we must return to (69). At $x_2 = 0$, the input end of the second line, we can establish from Fig. 6.13

$$V(x_2 = 0, p) = D = V(x_1 = s, p) - pLI(x_1 = s, p) \quad (78)$$

and actually combine the individual terms of the two traveling wave expansions so as to obtain, term by term, the contributions to D . We observe in accordance with (51) and (52) that the current terms have

identical forms as the voltage terms but have all positive signs and are divided by the factor $Z_c = R_{s1}$ for the first line. We also observe that

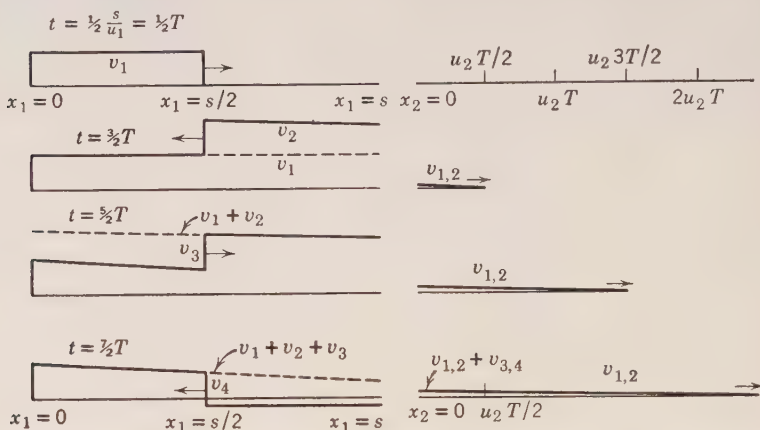


Fig. 6.14. Voltage distribution on two lossless transmission lines connected through a series inductance and with step voltage applied at $x_1 = 0$.

at $x_1 = s$ the first two terms in (73) need to be considered at once. If we designate therefore the first contribution to D as $D_{1,2}$ we obtain

$$D_{1,2}(p) = \frac{V_m}{p} \left[\left(1 - p \frac{L}{R_{s1}} \right) + \frac{p - \beta}{p + \alpha} \left(1 + p \frac{L}{R_s} \right) \right] e^{-p \frac{s}{u_1}} \quad (79)$$

or by combination and recall of (72)

$$V_{1,2}(x_2, p) = V_m \frac{\alpha - \beta}{p(p + \alpha)} e^{-p \frac{s}{u_1}} e^{-p \frac{x_2}{u_2}} \quad (80)$$

where we use the corresponding indices for the first outgoing voltage wave on the second line. The inverse transform of the rational fraction is found very readily from Table 1.3 and adjusting for the delay factor, we have

$$v_{1,2}(x_2, t) = \frac{\alpha - \beta}{\alpha} V_m [1 - e^{-\alpha(t - T - \theta)}] \quad t \geq (T + \theta) \quad (81)$$

where $\theta = x_2/u_2$ is the travel time along the second line. We see that at any point x_2 the voltage will rise exponentially with the same time constant $1/\alpha$ as in the first line to a final value

$$\frac{\alpha - \beta}{\alpha} V_m = \frac{2R_{s2}}{R_{s1} + R_{s2}} V_m$$

which is in fact the resultant final voltage at the end of the first line if we take only the first two traveling waves. In Fig. 6.14, the voltage

distribution along the second line is given by (81) alone at the instants $t = 3T/2$ and $5T/2$; obviously, the inductance has reduced the wave front entering the second line most effectively.

We can compute the second outgoing voltage wave by the same method as before and find first for the contribution $D_{3,4}(p)$ in accordance with (78) the value

$$D_{3,4}(p) = \frac{V_m}{p} \left\{ - \left(\frac{p - \beta}{p + \alpha} \right) \left(1 - p \frac{L}{R_{s1}} \right) - \left(\frac{p - \beta}{p + \alpha} \right)^2 \left(1 + p \frac{L}{R_{s1}} \right) \right\} e^{-p \frac{3s}{u_1}}$$

or by combination and use of (72) for the voltage contribution

$$V_{3,4}(p) = -V_m \cdot \frac{p - \beta}{p + \alpha} \cdot \frac{\alpha - \beta}{p(p + \alpha)} e^{-p \frac{3s}{u_1}} e^{-p \frac{x_2}{u_2}} \quad (82)$$

The inverse Laplace transform is best found by means of the sum of residues as stated in (1.57) with (1.69) for the second-order pole at $p = -\alpha$. The final result is

$$v_{3,4}(x_2, t) = \frac{\alpha - \beta}{\alpha} \left[\frac{\beta}{\alpha} [1 - e^{-\alpha(t-3T-\theta)}] - (\alpha + \beta)(t - 3T - \theta) e^{-\alpha(t-3T-\theta)} \right] \quad t \geq (3T + \theta) \quad (83)$$

where again $\theta = x_2/u_2$, the travel time along the second line. At the instant $t = \frac{3}{2}T$, this second voltage wave has progressed to $x_2 = \frac{1}{2}u_2T$ and has subtracted from the first wave (81) which has gone out to $x_2 = \frac{5}{2}u_2T$.

The inductance required for effective protection of a line against surges, such as demonstrated in the example for the second line, is generally impractical. A better arrangement is an inductance with shunt resistance to actually absorb energy; in such case the constants become more involved but the computational difficulties are not increased. One can also use a shunt capacitance rather than a series inductance which leads to the same results as obtained in the foregoing. A very extensive presentation of reflection and deformation of traveling waves on practical transmission lines with many experimental results is given in Bewley.^{C1}

6.9 Standing Wave Analysis

As we emphasized before, the expansions of the Laplace transform solutions (48) and (49) for the lossless line into the infinite series of

exponential functions give *exact* solutions by means of very few terms very shortly after the source has been applied to the line. This solution appears in terms of the traveling waves generated by the successive reflections at the terminals of the line. On the other hand, if we are interested in the long-term behavior of the transmission line, this traveling wave method becomes unwieldy.

We can, however, take the solutions in the closed forms (48) and (49) and apply to these directly the residue method of evaluating the inverse Laplace transform. For step source functions we can use the form (1.57) which gives the so-called indicial admittance or transmittance for (49) and (48), respectively; for general source functions it is simpler to use the form (1.55). Should higher order poles arise, then the residues at such poles must be computed in accordance with (1.69).

To illustrate the method, let us first choose the simple problem of a finite lossless line short-circuited at the far end with a voltage $v_0(t)$ applied directly to the input terminals, so that

$$Z_t = Z_0 = 0, \quad \rho_0 = \rho_t = 1$$

The voltage transform is therefore from (48) very simply

$$V(x, p) = \frac{e^{-nx} - e^{-n(2s-x)}}{1 - e^{-n2s}} \mathcal{L}v_0(t) = \frac{\sinh \frac{p}{u}(s-x)}{\sinh \frac{p}{u}s} \mathcal{L}v_0(t) \quad (84)$$

where the last form can be recognized if we multiply numerator and denominator of the exponential fraction by e^{ns} and recall from (50) that for the lossless line $n_0 = p/u$. We are thus concerned with a *meromorphic* function of p , such as we had encountered at the end of section 3.2. As outlined there, we may expand into partial fractions and define the sum of the residues as the inverse Laplace transform even though it might be an infinite series. To accept the final solution, we must demonstrate that any finite sum of the residues exists and converges uniformly in any finite interval of time towards a definite value as the number of terms in the series approaches infinity.

We may carry through the partial fraction expansion without as yet specifying the source voltage. To find the location of the poles, we admit for p complex values $\alpha + j\beta$. We then find the root values of the denominator in (84) from

$$\sinh(\alpha + j\beta) \frac{s}{u} = \sinh\left(\alpha \frac{s}{u}\right) \cosh\left(\beta \frac{s}{u}\right) + j \cosh\left(\alpha \frac{s}{u}\right) \sinh\left(\beta \frac{s}{u}\right) = 0$$

Here, both the real and the imaginary parts must be independently equal to zero which can be achieved only if we choose

$$\left(\alpha \frac{s}{u}\right) = 0 \quad \left(\beta \frac{s}{u}\right) = n\pi \quad n = 0, \pm 1, \pm 2, \dots$$

or also

$$p_n = jn\pi \frac{u}{s} \quad n = 0, \pm 1, \pm 2, \dots \quad (85)$$

The poles are thus located along the imaginary axis; they are all simple and occur in conjugate pairs, symmetrical about the pole $p = 0$ which also is simple. The actual expansion of the hyperbolic fraction gives in accordance with the general expression (1.54) with (1.41)

$$\begin{aligned} \frac{N(p)}{D(p)} &= \frac{\sinh \frac{p}{u} (s-x)}{\sinh \frac{p}{u} s} = \sum_{n=-\infty}^{+\infty} \frac{\sinh \left(p_n \frac{s-x}{u} \right)}{\frac{s}{u} \cosh \left(p_n \frac{s}{u} \right)} \frac{1}{p - p_n} \\ &= -j \frac{u}{s} \sum_{n=-\infty}^{+\infty} \frac{\sin n\pi \frac{x}{s}}{p - p_n} \quad (86) \end{aligned}$$

or if we introduce this into the voltage transform (84)

$$V(x, p) = -j \frac{u}{s} \sum_{n=-\infty}^{+\infty} \sin \left(n\pi \frac{x}{s} \right) \frac{\mathcal{L}v_0(t)}{p - p_n} \quad (87)$$

We now recognize (87) as the resolution of the voltage distribution into a spatial Fourier series with amplitudes which are functions of time, dependent only upon the nature of the source function! The fundamental wave length for the line distribution is defined by $n = 1$ as twice the length of the line or $\lambda_1 = 2s$.

For an applied step voltage $v_0(t) = V_m 1$ with the Laplace transform V_m/p , we need the inverse transform

$$\mathcal{L}^{-1} \frac{1}{p(p - p_n)} = -\frac{1}{p_n} (1 - e^{p_n t})$$

which we read from Table 1.3. Introducing p_n from (85), we can construct the inverse transform of (87) as

$$v(x,t) = \mathcal{L}^{-1}V(x,p) = V_m \sum_{n=-\infty}^{+\infty} \sin\left(n\pi \frac{x}{s}\right) \frac{1}{n\pi} (1 - e^{jn\pi \frac{u}{s}t}) \quad (87a)$$

To simplify, we note that for $n = 0$ no contribution results because the parenthesis vanishes and $(\sin x)/x$ remains finite for $x \rightarrow 0$. Combining the pairs of terms for $\pm|n|$, we need again only consider the parenthesis and so obtain as sum over the positive integers

$$2 - 2 \cos\left(n\pi \frac{u}{s}t\right)$$

Finally, separating the part independent of time, we have

$$v(x,t) = \frac{2}{\pi} V_m \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(n\pi \frac{x}{s}\right) - \frac{2}{\pi} V_m \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(n\pi \frac{x}{s}\right) \cos\left(n\pi \frac{u}{s}t\right) \quad (88)$$

The first sum constitutes the time average distribution of the voltage along the line and as a Fourier series represents a linear decrease* from $+V_m$ at $x = 0^+$ to zero value at $x = s$. The second sum is identical in all respects with the first but has each harmonic multiplied by a harmonic time function, so that each space harmonic as a standing wave oscillates with the appropriate multiple of the fundamental frequency about its time average distribution.

We obtain a slightly different conception of (88) if we apply to the trigonometric products in the second sum the expansion into sum and difference arguments, so that we can write this summation

$$\frac{1}{\pi} V_m \sum_{n=1}^{\infty} \left(\frac{1}{n} \sin \frac{n\pi}{s} (x - ut) + \frac{1}{n} \sin \frac{n\pi}{s} (x + ut) \right) \quad (89)$$

In this form we can identify each individual sum as an exact replica of the first sum in (88) of half the amplitude and shifted, respectively, a distance $\pm ut$ which is proportional to time and actually indicates a shift with the same speed as the traveling waves in section 6.5. However, the interpretation must be quite different! Whereas the representation in section 5.6 describes the establishment of the voltage along the line in terms of the actual physical phenomena verifiable by

* See, e.g., E. A. Guillemin, *The Mathematics of Circuit Analysis*, p. 464, Wiley, New York, 1949.

oscillograms, representation (88) with the modification (89) is purely mathematical. To see this clearer, let us abbreviate the time independent sum in (88) by $G(x)$; then we write

$$v(x,t) = G(x) - [\tfrac{1}{2}G(x - ut) + \tfrac{1}{2}G(x + ut)] \quad (90)$$

This is represented in Fig. 6.15, where b gives the stationary distribution $G(x)$ which is an infinitely repetitive sawtooth pattern; and where

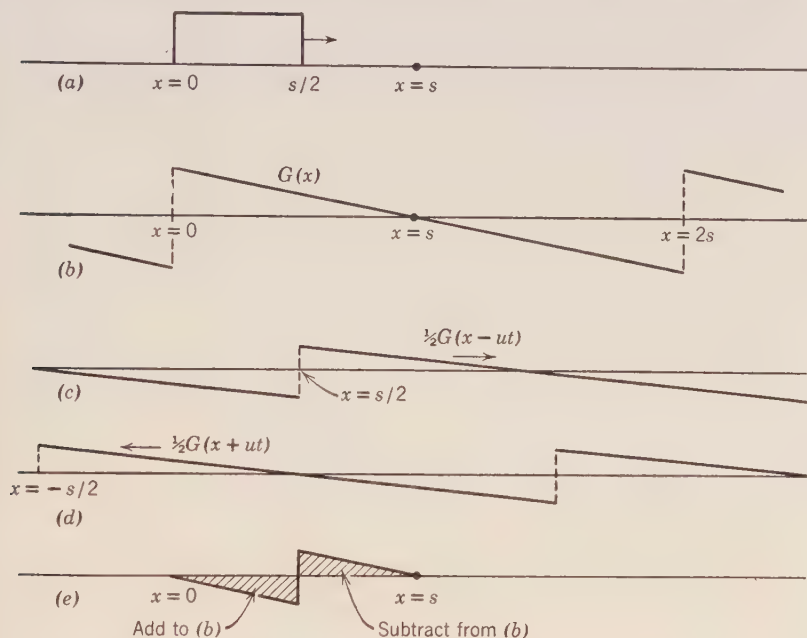


Fig. 6.15. Voltage distribution along a finite lossless transmission line by the standing wave method or d'Alembert method of solution. (a) Traveling wave at instant $t = T/2$, which equals the difference of (b) and (e).

c and d give the two identical infinitely periodic patterns of half amplitudes shifting in opposite directions with constant speed u and causing periodic fluctuations at any point x . The figure shows their instantaneous positions at $t = s/(2u) = T/2$ and e gives their sum which is the expression in brackets in (90). Subtracting the values e from b , the stationary pattern, leads exactly to a , the first step voltage wave having progressed over half the length of the line. The method of solution illustrated in (90) was first developed by d'Alembert and is usually named after him. The advantage of this form (90) becomes very obvious if we were to introduce $t = s/(2u)$ into (88) and to try the actual computation of the wave shape.

A rather different response is found for the current. From (49) we see that we only need to change the sign of the second exponential in the numerator of (84) and divide by R_s to obtain the current transform, so that

$$I(x, p) = \frac{e^{-nx} - e^{-n(2s-x)}}{1 - e^{-2ns}} \frac{\mathcal{L}v_0(t)}{R_s} = \frac{\cosh \frac{p}{u}(s-x)}{\sinh \frac{p}{u}s} \frac{\mathcal{L}v_0(t)}{R_s} \quad (91)$$

We can expand into partial fractions as in (86) and find because of the different numerator

$$I(x, p) = \frac{u}{s} \sum_{n=-\infty}^{+\infty} \cos\left(n\pi \frac{x}{s}\right) \frac{\mathcal{L}v_0(t)}{(p - p_n)R_s}$$

If we apply the same step voltage $v_0(t) = V_m 1$, we can proceed as for the voltage solution and obtain with (85) now

$$i(x, t) = \frac{V_m}{R_s} \sum_{n=-\infty}^{+\infty} \cos\left(n\pi \frac{x}{s}\right) \frac{1}{-jn\pi} (1 - e^{jn\pi \frac{u}{s}t}) \quad (92)$$

Here, however, we obtain a contribution from the term $n = 0$. We observe, since $\cos(0) = 1$

$$\lim_{n \rightarrow 0} \frac{1}{-jn\pi} \left(1 - \cos n\pi \frac{u}{s}t - j \sin n\pi \frac{u}{s}t\right) = \frac{u}{s}t$$

Combining the pairs of terms for $\pm|n|$ in (92), we now need to consider also the factor in front of the parenthesis, so that

$$\frac{1}{-jn\pi} (1 - e^{jn\pi \frac{u}{s}t}) + \frac{1}{jn\pi} (1 - e^{-jn\pi \frac{u}{s}t}) = \frac{2}{n\pi} \sin\left(n\pi \frac{u}{s}t\right)$$

The complete expression for the current is thus

$$i(x, t) = \frac{V_m}{R_s} \frac{u}{s}t + \frac{2}{\pi} \frac{V_m}{R_s} \sum_{n=1}^{\infty} \frac{1}{n} \cos\left(n\pi \frac{x}{s}\right) \sin\left(n\pi \frac{u}{s}t\right) \quad (93)$$

The first term is only a function of time and rises indefinitely. It contains no propagation effect whatsoever, in fact is a unique function of time for all points along the line. The second term is the identical sum as in (88) except for the factor $1/R_s$. We verify this readily by

expanding the trigonometric product which leads to the same result as (89).

To check the correctness of (93) let us apply it at $x = 0$, for which

$$i(0,t) = \frac{V_m}{R_s} \left[\frac{u}{s} t + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \left(n\pi \frac{u}{s} t \right) \right]$$

The sum is now identical with $(1/R_s)G(ut)$ as comparison with (88) indicates, so that it represents the sawtooth pattern as in Fig. 6.15 with ut the travel distance x along the line. If added to the linear rise t/T given by the first term, we obtain a staircase function with a first step of value V_m/R_s and rising in abrupt steps at every interval $2T$ by the value $2V_m/R_s$. This is exactly the shape of the input current for the short-circuited lossless line as it builds up by the repeated positive reflections at both terminal ends of the line. For any point $x \neq 0$, we need to consider the local shift as given by (89) and we shall find the appropriate local staircase pattern.

These simple examples served to clarify the essential aspects of the standing wave method. Lest there be unwarranted optimism that we always can find such simple interpretations, let us find for this same lossless line the voltage distribution if a sinusoidal source voltage is applied

$$v_0(t) = V_m \sin \omega t, \quad \mathcal{L}v_0(t) = \text{Im} \frac{V_m}{p - j\omega}$$

As we illustrated elsewhere, we can save considerable labor if we use the complex notation throughout, i.e., introduce

$$\bar{V}_0(p) = \frac{V_m}{p - j\omega}$$

and then in (87) write

$$\bar{V}(x,p) = -j \frac{u}{s} V_m \sum_{n=-\infty}^{+\infty} \sin \left(n\pi \frac{x}{s} \right) \frac{1}{(p - j\omega)(p - p_n)} \quad (94)$$

where the bar is the warning that in the end we need to take only the imaginary part of both sides to obtain the real physical voltage solution. We find the inverse Laplace transform of (94) by taking the sum of residues for the last fraction

$$\mathcal{L}^{-1} \frac{1}{(p - j\omega)(p - p_n)} = \frac{e^{j\omega t}}{j\omega - p_n} + \frac{e^{p_n t}}{p_n - j\omega}$$

so that with (85)

$$\bar{v}(x, t) = \frac{u}{s} V_m \sum_{n=-\infty}^{+\infty} \sin\left(n\pi \frac{x}{s}\right) \frac{1}{n\pi(u/s) - \omega} (e^{j\omega t} - e^{jn\pi \frac{u}{s} t})$$

If we let $\omega \rightarrow 0$ here we get at once the previous form for the applied step voltage (87a). Simplifying as there, we note that for $n = 0$ we have no contribution. Combining the pairs of terms for $\pm|n|$ and taking only the imaginary parts of the result, we finally arrive at

$$\begin{aligned} v(x, t) = & 2V_m \left[\sum_{n=1}^{\infty} \frac{n\pi}{(n\pi)^2 - [\omega(s/u)]^2} \sin\left(n\pi \frac{x}{s}\right) \right] \sin \omega t \\ & - 2V_m \left[\sum_{n=1}^{\infty} \frac{\omega(s/u)}{(n\pi)^2 - [\omega(s/u)]^2} \sin\left(n\pi \frac{x}{s}\right) \sin\left(n\pi \frac{u}{s} t\right) \right] \quad (95) \end{aligned}$$

which certainly does not permit as simple an interpretation as for the applied step voltage, even though we have the clear separation of the steady-state a-c distribution from the superimposed transient response.

If we take a distortionless line instead of a lossless line, the only change is the replacement of $n_0 = p/u$ by $n_d = (p + \delta)/u$ as shown in (35). To demonstrate the effect of the attenuation, we might evaluate the location of the poles for the short-circuited line for which we can use directly (84), replacing in the hyperbolic fraction p by $p + \delta$. Since the root values of $\sinh(\alpha + j\beta)(s/u) = 0$ must remain the same, this results in the shift of the location of the poles to

$$p_n' = -\delta + jn\pi \frac{u}{s} \quad n = 0, \pm 1, \pm 2 \cdots \quad (96)$$

so that actually the only change in (86) occurs in the linear fractions where p_n is replaced by p_n' . We can therefore write directly for (87) here

$$V(x, p) = -j \frac{u}{s} \sum_{n=-\infty}^{\infty} \sin\left(n\pi \frac{x}{s}\right) \frac{\mathcal{L}v_0(t)}{p - p_n'} \quad (97)$$

This leads for the step voltage source to the inverse transform

$$\mathcal{L}^{-1} \frac{1}{p(p - p_n')} = \frac{1}{-p_n'} (1 - e^{p_n' t})$$

The final inverse transform of (97) does not permit the simplifications

possible in the previous cases. Proceeding as for (88), we find here finally

$v(x, t)$

$$= \frac{2}{\pi} V_m \sum_{n=1}^{\infty} \frac{n}{n^2 + [(\delta/\pi)(s/u)]^2} \sin\left(n\pi \frac{x}{s}\right) \left[1 - e^{-\delta t} \cos\left(n\pi \frac{u}{s} t\right)\right] \\ - \frac{2}{\pi} V_m \sum_{n=1}^{\infty} \frac{(\delta/\pi)(s/u)}{n^2 + [(\delta/\pi)(s/u)]^2} \sin\left(n\pi \frac{x}{s}\right) e^{-\delta t} \sin\left(n\pi \frac{u}{s} t\right) \quad (98)$$

The first line becomes identical with (88) if we set $\delta = 0$, whereas the second line vanishes in that case. The effect of the line attenuation is therefore threefold: the sustained oscillations in the lossless line about the average distribution become damped and eventually die out; the final distribution has smaller amplitudes of the higher space harmonics than the average distribution in the lossless line; and a dephasing of the damped oscillations takes place caused by the losses. Actual numerical computations are extremely laborious unless the attenuation is very high.

The standing wave method will have distinct advantage when the initial conditions include a given current or voltage (charge) distribution along the line as it might be induced by thunder clouds or lightning strokes. An extensive treatment with experimental corroboration is found in Bewley,^{C1} chapter 11.

PROBLEMS

6.1 A lossless finite transmission line is terminated into a load resistance of one-third the value of the characteristic impedance of the line. If one applies a unit step voltage at the sending end, when will the current at the load be 90% of its final value?

6.2 A distortionless, infinitely long smooth line is suddenly connected to unit step voltage through a series condenser C . What is the entering current into the line?

6.3 A lossless transmission line is terminated into a series combination of inductance L and capacitance C . Evaluate the first reflected wave if a step voltage $V_0 I$ is applied at the sending end.

6.4 A rectangular voltage pulse of magnitude V and duration T is applied to an infinite distortionless line over a terminal capacitance C in series with the line.

(a) Find the current oscillogram at a distance T/\sqrt{lc} from the sending end.

(b) Find the voltage oscillogram at the distance $2T/\sqrt{lc}$ from the sending end.

6.5 Find the current response at the end of a finite distortionless line if a voltage $V_m \cos \omega t$ is applied to its terminals and the far end terminated into a resistance $R = \frac{1}{2}R_s$. Assume $\exp(-\delta s/u) = 0.8$ and $\omega s/u = 0.4\pi$.

6.6 A single sawtooth pulse of voltage rising linearly from zero to V in T

seconds is applied to a distortionless line of parameters $l_1 = 0.002$ h/mile, $c_1 = 0.0154$ microfarad/mile, $r_1 = 0.10$ ohm/mile, and of length $s_1 = 20$ miles. This distortionless line has a capacitance C of 10 microfarads connected across the far end terminals and joins directly a lossless line of parameters $l_2 = 0.001$ h/mile, $c_2 = 0.030$ microfarad/mile and of infinite length. (a) Find the first pulse transmitted into the second line and compare its shape with the applied signal. (b) Find the first reflected pulse traveling back towards the sending end of the first line halfway down this line.

6.7 Applying a sinusoidal voltage for one period directly to the terminals of a distortionless line which is terminated at the far end into a capacitance C , what will be the first reflected voltage wave returning towards the sending end? You are not asked to give the complete solution, but a qualitative picture is desirable by means of Laplace transform analysis.

6.8 A distortionless transmission line of length s is connected to an infinitely long lossless line. The parameters of the first line are r_1 , g_1 , c_1 , and $l_1 = r_1 c_1 / g_1$; those of the second line l_2 and c_2 . A single half sine wave pulse is applied to the first line over a resistance of value $\sqrt{l_1 / c_1}$ by a generator of zero internal impedance. Find the oscillograms of the pulse taken on either side of the junction if the period of the sine wave is one-half the time of travel of signals along the first line. Assume also that the two characteristic impedances of the lines have the ratio $(Z_2 / Z_1) = 2$.

6.9 A lossless line of length s is connected to another infinite lossless line of the same parameters through an inductance L in series and a shunt resistance G which we might take joined at the center of L to make the lumped network symmetrical. (a) Find the first reflected voltage and current waves if a step voltage is applied at the input terminals of the first line. (b) Find the first transmitted voltage and current wave in the second line. (c) Draw the oscillograms for the same conditions as in section 6.8 and assume that G is equal to the total inverse resistance of the first line.

6.10 Take the same arrangement as in problem 6.9 but assume that the two lines are de-energized, and the first line short-circuited at the sending end. Assume further that the inductance carries an initial current i_0 at $t = 0$ and solve the current and voltage distribution with this initial condition.

6.11 Take an infinite distortionless line and assume that initially a voltage impulse $M_v S_0(t)$ is applied at a distance s from the sending end. The lines might be de-energized and the impulse might be caused by lightning. From initial symmetry we can assume that one-half of this voltage impulse will travel in the opposite directions. Describe the phenomenon of reflection of both current and voltage.

6.12 A lossless line is terminated at both ends into its surge resistance. The far end has a sudden change in load, so that its terminal resistance drops to one-half the original value. The generator at the sending is assumed to have zero internal impedance and furnishes a sinusoidal voltage $V_m \sin \omega t$. Describe the current and voltage adjustment to the changed termination in terms of traveling waves.

6.13 Take the same conditions as in problem 6.12 but describe the adjustment in terms of standing waves.

6.14 Give the standing wave solution for a distortionless line terminated at both ends into half its surge resistance and supplied at the sending end with a sinusoidal voltage $V_m \sin \omega t$.

6.15 A lossless line of length s is terminated into a series combination of inductance and resistance, where $L = ls$, and $R = R_s$. Obtain the standing wave solution for applied sinusoidal voltage $V_m \cos \omega t$.

6.16 A finite distortionless line is terminated at the far end into an inductance L . A step voltage $V_m i$ is applied directly to the sending end terminals. (a) Find the voltage distribution by the standing wave method. (b) Find the current distribution by the standing wave method. For numerical calculation assume that the value of the inductance equals one-tenth the value of the total line inductance.

7. NONINDUCTIVE CABLES

Comparatively short lengths of uniform electric cable with good insulation properties can be sufficiently characterized by only two parameters, resistance and capacitance per unit length. The resulting differential equations and their solutions were first encountered in problems of diffusion of heat through materials; they are therefore referred to as *diffusion equations*. They frequently come up also in molecular diffusion problems in physical chemistry. As a class of phenomena, their properties deserve special discussion as distinct from the phenomenon of wave propagation dealt with in chapter 6.

7.1 The Infinitely Long Cable; Entering Current

Having given the general transmission-line equations and their solutions in sections 6.2 and 6.3, we need only set these down here for the special case that $l = g = 0$. This means that we will disregard leakance, which is justified for good insulation, and inductance, which is only feasible for comparatively short lengths of cable transmission. The Laplace transforms of the general differential equations (6.21) reduce to

$$\begin{aligned} -\frac{\partial V(x,p)}{\partial x} &= rI(x,p) \\ -\frac{\partial I(x,p)}{\partial x} &= cpV(x,p) - cv(x,0) \end{aligned} \tag{1}$$

where $v(x,0)$ is the initial voltage distribution on the cable which must be given as part of the specific problem; we will usually disregard this initial term. Without $v(x,0)$ the two first-order differential equations can be combined by, e.g., differentiating the first one again with respect to x and introducing for the current transform derivative the second line, so that

$$\frac{\partial^2 V(x,p)}{\partial x^2} = rcpV(x,p) = n^2 V(x,p) \tag{2}$$

which is identical in form to (6.26) with the simplified version of the parametric propagation function n . The general solution of (2) in terms of the distance variable x is identical with (6.27), namely

$$V(x, p) = Ae^{nx} + Be^{-nx} \quad (3)$$

and from the first line of (1) we find at once the solution for the current transform as

$$I(x, p) = -\frac{n}{r}(Ae^{nx} - Be^{-nx}) = \frac{1}{Z_c}(-Ae^{nx} + Be^{-nx}) \quad (4)$$

where the characteristic impedance is defined in analogy to (6.30) by

$$Z_c = \frac{r}{n} = \sqrt{\frac{r}{cp}} \quad (5)$$

We note at once two most important differences between this cable and the distortionless lines. The propagation function from (2)

$$n = \sqrt{rcp} \quad (6)$$

is irrational here, and was rational for the lossless line (6.33) as well as the distortionless line (6.35). The characteristic impedance (5) is also irrational here but was a fixed constant in (6.38). We must deal, therefore, with algebraic functions of a more general type and, in

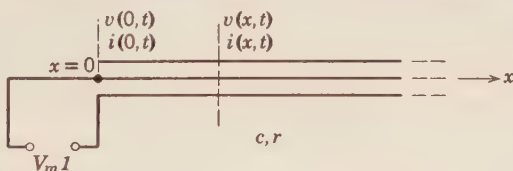


Fig. 7.1. Infinitely long noninductive cable.

particular, must anticipate a more cautious approach to the evaluation of the inverse Laplace transform which involves integration in the complex p -plane.

As the simplest approach, let us first consider the infinitely long ideal cable symbolized in Fig. 7.1 and apply to its input terminals at $x = 0$ directly a step voltage $v_0(t) = V_m I$ with the Laplace transform V_m/p . In the definition of the inverse Laplace transform we have stipulated $\text{Re}(p) > 0$; if we define p as a complex variable in the polar form

$$p = |p|e^{j\varphi} \quad -\pi < \varphi < +\pi \quad (7)$$

the path of integration must be restricted to $-\pi/2 < \varphi < +\pi/2$. On

the other hand, the double-valued function \sqrt{p} covers the respective ranges

$$\sqrt{p} = \begin{cases} |p|^{\frac{1}{2}} e^{j\varphi/2} & \text{Re } \sqrt{p} > 0, \text{ branch 1} \\ |p|^{\frac{1}{2}} e^{j(\varphi/2+\pi)} & \text{Re } \sqrt{p} < 0, \text{ branch 2} \end{cases} \quad (8)$$

where the second branch requires a second leaf of the p -plane with $\pi < \varphi < 3\pi$ in order to establish a complete one-to-one correspondence between the complete \sqrt{p} -plane and the reservoir of p -values.

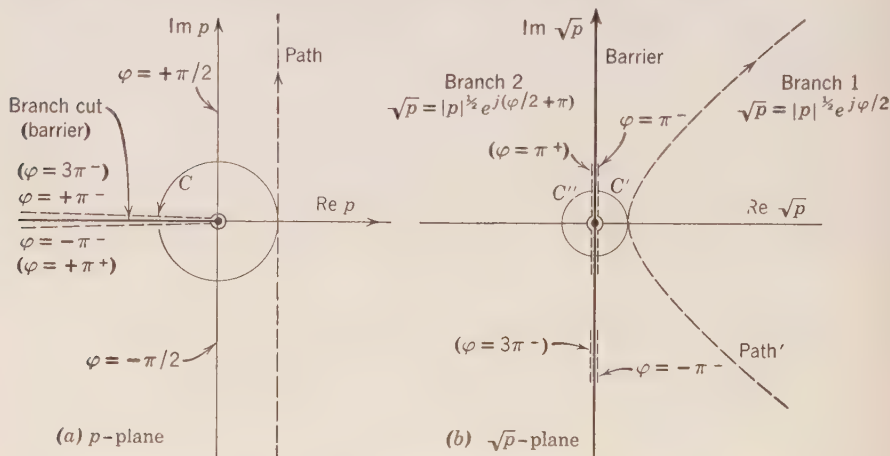


Fig. 7.2. Definition of the analytic branches for the algebraic function \sqrt{p} .
(a) p -plane; (b) \sqrt{p} -plane.

This is pictorially indicated in Fig. 7.2 where the left-hand side gives the p -plane with the circular arc C extending from $\varphi = -\pi^-$ (a little less than $-\pi$) to $\varphi = +\pi^-$ (a little less than $+\pi$), leaving a gap across $\varphi = \pi$. The right-hand side of this figure gives the p -plane in which branch 1 is the right-half plane with C' the image of the circular arc C in the p -plane. As a complex variable in its own right, \sqrt{p} should, however, cover the entire \sqrt{p} -plane. This can be done as in (8) by providing for the \pm sign of the square root in terms of $e^{j\pi}$, or it can be done by providing a second leaf of the p -plane with $\pi < \varphi < 3\pi$ and joined along $(\varphi = \pi^+)$ across the gap to $\varphi = \pi^-$ of the first leaf. The radius vector $\varphi = \pi$ becomes then the natural branch cut or *barrier* of the p -plane, separating in the \sqrt{p} -plane the two branches as the imaginary axis. As long as we move in one or the other leaf of the p -plane we have a unique one-to-one relationship between any p -point and its image in the \sqrt{p} -plane. We must avoid getting right unto the

line $\varphi = \pi$ because that is common to both leaves; once we are on $\varphi = \pi$, we would have a choice to continue in either leaf! To retain absolute one-to-one correspondence, it is therefore *necessary to exclude the barrier* or branch cut $\varphi = \pi$ from either leaf, i.e., to leave a gap in the two p -plane leaves around $\varphi = \pi$. The circular arc C then has a second image in the \sqrt{p} -plane, the arc C'' from the image ($\varphi = \pi^+$) without actually closing into a circle.

The path of integration for the inverse Laplace transform is designated as *Path* in the p -plane. Should it be located in the leaf $-\pi < \varphi < \pi$? If so, then its image in the \sqrt{p} -plane is given by the dotted *Path'* in the right-half plane along which $\text{Re}(\sqrt{p}) > 0$. This will assure in (6)

$$\text{Re}(n) = \text{Re}(\sqrt{rc} \sqrt{p}) > 0$$

and thus requires that we choose as a possible solution from (3) the *negative* exponential in order to provide for convergence of the inverse Laplace integral! This is exactly the same choice we made in section 6.4 for the lossless and distortionless lines.

To introduce the boundary condition, we note that at $x = 0$ the Laplace transform of the line voltage must be identical with that of the source voltage, so that

$$V(0, p) = B e^{-nx} \big|_{x=0} = B = \frac{V_m}{p} = \mathcal{L}v_0(t) \quad (9)$$

The general solution thus becomes

$$V(x, p) = \mathcal{L}v_0(t) e^{-nx} = \frac{V_m}{p} e^{-\sqrt{rcp}x} \quad \text{Re } \sqrt{p} > 0 \quad (10)$$

where we indicated also the proper choice of the branch of the two-valued function.

We find for the current from (4) with (5) in similar notation

$$I(x, p) = \frac{\mathcal{L}v_0(t)}{Z_c} e^{-nx} = \frac{V_m}{\sqrt{r/c}} \frac{1}{\sqrt{p}} e^{-\sqrt{rcp}x} \quad \text{Re } \sqrt{p} > 0 \quad (11)$$

Let us first determine the inverse Laplace transform of the current entering the line at $x = 0$, for which

$$I(0, p) = \frac{V_m}{\sqrt{r/c}} \frac{1}{\sqrt{p}} \quad \text{Re } \sqrt{p} > 0 \quad (12)$$

By definition the inverse Laplace transform is from section 1.4

$$i(0,t) = \frac{V_m}{\sqrt{r/c}} \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{1}{\sqrt{p}} e^{pt} dp \quad \text{Re } \sqrt{p} > 0 \quad (13)$$

In general, we have evaluated this complex integral actually by means of Cauchy's integral theorem, substituting a closed path and taking the sum of residues. To validate this procedure we showed that the contributions of all other sections of the closed path were vanishingly small. This can be readily done for analytic functions where Cauchy's theorem applies directly. In (13), however, we have to deal with an algebraic function so that we first must reduce it to analyticity in the p -plane. Referring to Fig. 7.2a, each of the two leaves of the p -plane defines an analytic branch of the function \sqrt{p} , as long as we exclude the neighborhood of $\varphi = \pi$ which is the *barrier* or *branch cut*. We have already specified branch 1 of \sqrt{p} related to the leaf $-\pi < \varphi < +\pi$ of the p -plane in one-to-one correspondence as defining the physically possible solution of the problem. If we choose, then, a closed path in the p -plane as shown in Fig. 7.3, we can apply to it the Cauchy integral theorem in the identical manner as for the rational functions dealt with in Vol. I, chapter 5. We shall therefore again evaluate the desired integral (13) with the path C' by first finding the value of the closed path integral and then subtracting all the sectional contributions to leave what we need. Obviously, in doing this here in considerable detail we will establish some general results for similar problems.

The integrand in (13) has as the only singularity in the finite p -plane a branch point at $p = 0$, which we have already made part of the branch cut $\varphi = \pi$ and thus excluded from the region of definition of the integrand. The integral over the closed contour $ABDE'G'G''E''FA$ in Fig. 7.3 will therefore give zero result, so that we could also write

$$\begin{aligned} & \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{1}{\sqrt{p}} e^{pt} dp \\ &= \frac{1}{2\pi j} \left(\int_{AF} + \int_{FE''} + \int_{E''G''} + \int_{G''G'} + \int_{G'E'} + \int_{E'D} + \int_{DB} \right) \quad (14) \end{aligned}$$

i.e., define the two paths as equivalent in the Cauchy sense if we let $R \rightarrow \infty$, and for convenience also $\rho \rightarrow 0$. The demonstration that the contribution to the integral (14) of the vertical sections of path C' below A and above B are vanishingly small for R very large follows the pattern in Vol. I, section 5.6. We set $p = c + j\omega$, $dp = j d\omega$ since only ω varies along the vertical path C' and for R large we approximate

$\sqrt{p} \approx \sqrt{j\omega}$. The integral beyond B becomes

$$\frac{\sqrt{j}}{2\pi j} e^{ct} \int_R^\infty \frac{e^{j\omega t}}{\sqrt{\omega}} d\omega = \frac{e^{ct}}{2\pi \sqrt{j} \sqrt{t}} \int_{\mu=Rt}^\infty \frac{e^{j\mu}}{\sqrt{\mu}} d\mu \quad (15)$$

where the new variable $\mu = \omega t$ was introduced. This brings the integral into the complex form of the Fresnel integral* which is the

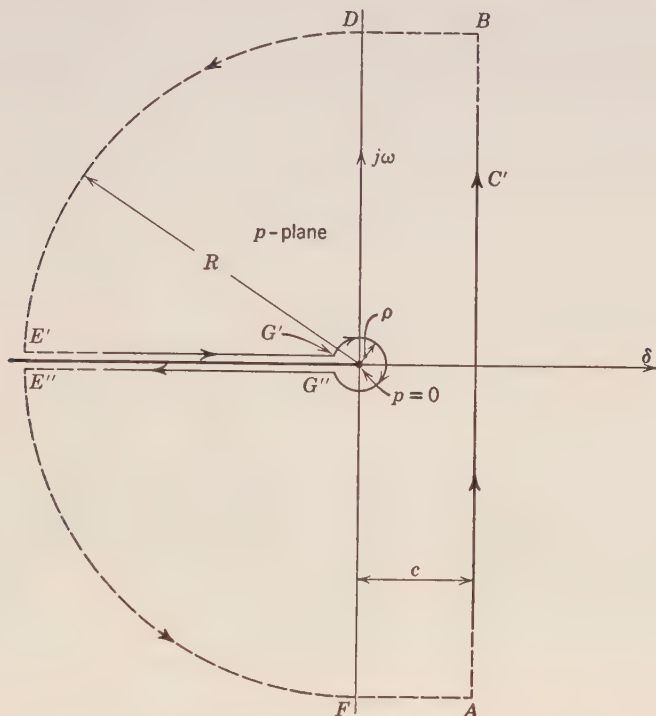


Fig. 7.3. Analytic path of integration for the entering current (13).

complex combination of tabulated real Fresnel integrals occurring in diffraction problems. With R chosen suitably large, (15) and similarly the corresponding integral along C' below A can be made as small as we please, so that we can restrict our attention fully to the integrals in (14).

For the horizontal section DB we set $p = \delta + jR$, $dp = d\delta$, and $\sqrt{p} \approx \sqrt{j} \sqrt{R}$ since δ can at most take the value c and $c \ll R$. The

* E. Jahnke and F. Emde, *Tables of Functions*, 3rd edition, p. 36, B. G. Teubner, Leipzig, 1938. Also reprinted by Dover Publications, New York, 1943.

contribution in (14) can therefore be delimited

$$\left| \frac{1}{2\pi j} \frac{1}{\sqrt{j}} \frac{1}{\sqrt{R}} e^{jRt} \int_{\delta=0}^{\delta=c} e^{\delta t} d\delta \right| \leq \frac{1}{2\pi \sqrt{R}} c e^{ct} \quad (16)$$

because the absolute values of $j^{3/2}$ and e^{jRt} will be unity and the integral will certainly be less than the maximum value of the integrand multiplied by the length of the path. Obviously, choosing R sufficiently large, we can make this contribution negligible and, of course, do the same at once for the integral along AF . By similar reasoning, as given in Vol. I, section 5.6 leading to (5.79), we can also show that the sectional integrals over the quarter circles FE'' and $E'D$ can be made vanishingly small.

This leaves then the integrals along the parallel horizontal lines and the small circle around the origin. For the latter we can write $p = \rho e^{j\varphi}$, $dp = j\rho e^{j\varphi} d\varphi$, $\sqrt{p} = \sqrt{\rho} e^{j\varphi/2}$, so that

$$\left| \frac{1}{2\pi j} \int_{\varphi=-\pi^-}^{\varphi=+\pi^-} \frac{1}{\sqrt{\rho}} e^{-j\varphi/2} e^{\rho \cos \varphi} e^{\rho j \sin \varphi} j \rho e^{j\varphi} d\varphi \right| < \frac{1}{2\pi} (e^{\rho} \sqrt{\rho}) \int_{-\pi}^{+\pi} d\varphi = e^{\rho} \sqrt{\rho} \quad (17)$$

If we again take the absolute values of all imaginary exponentials as unity, replace $\exp(\rho \cos \varphi)$ by e^{ρ} , then the desired integral must be definitely less than the one with the simplified integrand which just gives 2π . The final upper bound can be made as small as we please by choosing ρ very small.

For the integral along $E''G''$ we can take $p = |p|e^{-j\pi^-}$, recalling Fig. 7.2a for the lower boundary of the gap; this makes

$$\sqrt{p} = |p|^{1/2} e^{-j\pi/2} = \frac{1}{j} \sqrt{\mu}, \quad p = -\mu, \quad dp = -d\mu$$

if we define $|p| \cos \pi^- = -\mu$ and disregard the very slight imaginary component because it is not involved in the integration. The integral is now

$$\frac{1}{2\pi j} \int_{\mu=R}^{\mu=\rho} \frac{j}{\sqrt{\mu}} e^{-\mu t} (-d\mu) = \frac{1}{2\pi} \int_{\mu=\rho}^R e^{-\mu t} \frac{d\mu}{\sqrt{\mu}} = \frac{1}{2\pi \sqrt{t}} \int_{y=0}^{y \rightarrow \infty} e^{-y} \frac{dy}{\sqrt{y}} \quad (18)$$

where the last form uses the new variable $y = \mu t$. Since we can let ρ become very small and R very large, we can interpret the integral as

the standard form of the gamma function $\Gamma(\frac{1}{2})$, as evidenced by the general definition*

$$\Gamma(z) = (z - 1)! = \int_0^\infty e^{-y}y^{(z-1)} dy \tag{19}$$

Actually, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ as found also in the references, so that the contribution of (18) becomes $(\frac{1}{2})(\pi t)^{-1/2}$. Quite similarly, we can set for the integral along $G'E'$ for the variable

$$p = |p|e^{+j\pi^-} = -\mu, \qquad \sqrt{p} = |p|^{1/2}e^{-j\pi/2} = j\sqrt{\mu}, \qquad dp = -d\mu$$

so that we have

$$\frac{1}{2\pi j} \int_{\mu=\rho}^{\mu=R} \frac{1}{j\sqrt{\mu}} e^{-\mu t} (-d\mu) = \frac{1}{2\pi} \int_{\mu=\rho}^R e^{-\mu t} \frac{d\mu}{\sqrt{\mu}} \rightarrow \frac{1}{2\pi\sqrt{t}} \Gamma\left(\frac{1}{2}\right) \tag{20}$$

giving the identical contribution as (18).

The total result of all integrals in (14) is thus

$$\frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{1}{\sqrt{p}} e^{pt} dp = \frac{1}{\sqrt{\pi t}} \tag{21}$$

and the entering current from (13) is simply

$$i(0,t) = \frac{V_m}{\sqrt{r/c}} \frac{1}{\sqrt{\pi t}} \tag{21a}$$

The disregard of the inductance leads to a current of initially infinite value which decreases to zero value as the cable satiates with electric charge.

7.2 Response of the Infinite Cable to a Step Voltage

The complete Laplace transform solutions for current and voltage were found in (11) and (10), respectively. Let us now evaluate the inverse Laplace transform for the current

$$i(x,t) = \mathfrak{L}^{-1} \frac{V_m}{\sqrt{r/c}} \frac{1}{\sqrt{p}} e^{-\sqrt{rcp}x} = \frac{1}{2\pi j} \frac{V_m}{\sqrt{r/c}} \int_{c-j\infty}^{c+j\infty} \frac{1}{\sqrt{p}} e^{-\sqrt{rcp}x} e^{pt} dp \qquad \text{Re } \sqrt{p} > 0 \tag{22}$$

We note the earlier decision that $\text{Re } \sqrt{p} > 0$ designates the branch of the algebraic function we want to use. With this choice, we have already made the integrand in (22) an analytic function over the entire

* Jahnke-Emde, *op. cit.*, p. 20; also McLachlan,^{D13} pp. 72-77; and almost any book on advanced calculus.

leaf $-\pi < \varphi < +\pi$ of the p -plane because the integrand does not possess any singularity in the finite p -plane except the branch cut. This entitles us, then, to use the same path substitution as in (14) and, indeed, we can repeat all the arguments from there to demonstrate that we need only compute the sectional integrals along $E''G''$ and $E'G'$, and that all other sectional contributions can be made to vanish. We have in (22) the additional convergence factor $e^{-\sqrt{ap}}$ with $a = rcx^2$, which assists in the arguments because $\text{Re } \sqrt{p} > 0$.

Along the paths $E''G''$ and $G'E'$ we set again as in section 7.1 with reference to Fig. 7.2

$$\text{for } E''G'': \quad p = |p|e^{-j\pi} = -\mu, \quad \sqrt{p} = \frac{1}{j}\sqrt{\mu}, \quad dp = -d\mu \quad (23)$$

$$\text{for } E'G': \quad p = |p|e^{+j\pi} = -\mu, \quad \sqrt{p} = j\sqrt{\mu}, \quad dp = -d\mu$$

so that the sum of the two integrals can be contracted into

$$\begin{aligned} \frac{-1}{2\pi j} \int_{\infty}^0 \frac{1}{-j\sqrt{\mu}} e^{j\sqrt{a\mu}} e^{-\mu t} d\mu + \frac{-1}{2\pi j} \int_0^{\infty} \frac{1}{j\sqrt{\mu}} e^{-j\sqrt{a\mu}} e^{-\mu t} d\mu \\ = \frac{1}{\pi} \int_0^{\infty} \frac{\cos \sqrt{a\mu}}{\sqrt{\mu}} e^{-\mu t} d\mu \quad (24) \end{aligned}$$

This final form of the inverse Laplace transform of (22) actually has the same form as the direct Laplace integral in the real variable μ (instead of t) and the parameter t (instead of p) if we compare with (5.13). We can therefore find the value of the integral as a Laplace integral (or Fourier integral) from the Campbell-Foster tables,^{E1} pair 652.2, (see also section 7.4 below), namely

$$\mathcal{L} \left(\frac{1}{\sqrt{\pi\mu}} \cos 2\sqrt{\frac{\mu}{c}} \right) = \frac{1}{\sqrt{p}} \exp \left(-\frac{1}{cp} \right) \quad (25)$$

and if we appropriately transpose the result, replacing p by t and t by $4/a$, also taking $\sqrt{\pi}$ on the other side

$$\int_0^{\infty} \frac{1}{\sqrt{\mu}} \cos(\sqrt{a\mu}) e^{-\mu t} d\mu = \sqrt{\frac{\pi}{t}} \exp \left(-\frac{a}{4t} \right) \quad (26)$$

Thus the final result for (22) becomes

$$i(x, t) = \frac{V_m}{\sqrt{r/c}} \frac{1}{\sqrt{\pi t}} \exp \left(-\frac{rcx^2}{4t} \right) \quad t > 0 \quad (27)$$

For $x = 0$, this obviously reduces to the entering current expression (21a).

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To identify the integral in (24) we have made use of tables. Actually, we should have a method available which will lead to the same result but will permit us direct evaluation. We have demonstrated that (22) need be integrated only along $E''G''$ and $G'E'$. Since the function is analytic in the entire leaf $-\pi < \varphi < +\pi$ of the p -plane, we can apply to the troublesome exponential with the fractional power exponent in p the absolutely convergent Taylor series expansion

$$e^{-\sqrt{ap}} = 1 - \sqrt{ap} + \frac{1}{2!} ap - \frac{1}{3!} (ap)^{3/2} + \frac{1}{4!} (ap)^2 - \dots \quad (28)$$

so that

$$\begin{aligned} \frac{1}{\sqrt{p}} e^{-\sqrt{ap}} &= \left(1 + \frac{ap}{2!} + \frac{(ap)^2}{4!} + \dots + \frac{(ap)^n}{2n!} + \dots \right) \frac{1}{\sqrt{p}} \\ &\quad - \left(1 + \frac{ap}{3!} + \frac{(ap)^2}{5!} + \dots + \frac{(ap)^m}{(2m+1)!} + \dots \right) \sqrt{a} \end{aligned} \quad (29)$$

Here we collected the terms into two simpler series, each one of which is more readily integrable in the complex plane as required by (22).

In the second series of (29) containing the positive powers of p we find upon introduction of (23) that the typical integrals along $E''G''$ and $G'E'$

$$\frac{-1}{2\pi j} \int_{\infty}^0 (-\mu)^m e^{-\mu t} d\mu + \frac{-1}{2\pi j} \int_0^{\infty} (-\mu)^m e^{-\mu t} d\mu = 0$$

cancel each other as we see if we reverse the limits in the first integral. In the first series of (29), the typical integrals add because of the opposite sign of \sqrt{p} , so that

$$\begin{aligned} \frac{-1}{2\pi j} \int_{\infty}^0 (-\mu)^n \frac{j}{\sqrt{\mu}} e^{-\mu t} d\mu + \frac{-1}{2\pi j} \int_0^{\infty} (-\mu)^n \frac{1}{j\sqrt{\mu}} e^{-\mu t} d\mu \\ = \frac{(-1)^n}{\pi} \int_0^{\infty} \mu^{n-1/2} e^{-\mu t} d\mu \end{aligned}$$

If we set $\mu t = y$, we can identify from (19)

$$\int_0^{\infty} y^{(n+1/2)-1} e^{-y} dy = \Gamma(n + \frac{1}{2}) \quad (30)$$

and therefore the final sum for (22)

$$i(x, t) = \frac{V_m}{\sqrt{r/c}} \sum_{n=0}^{\infty} \frac{a^n}{2n!} \frac{(-1)^n}{\pi} \frac{\Gamma(n + \frac{1}{2})}{t^{n+1/2}} \quad (31)$$

This should, of course, be identical with the series development of the exponential in (27). If we recall the definition of the particular Γ function*

$$\Gamma\left(n + \frac{1}{2}\right) = \left(n - \frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2^n} [1 \cdot 3 \cdot 5 \cdots (2n - 1)] \quad (32)$$

and combine this with $2n!$ in (31), we observe cancellation of all odd factors, leading to

$$\frac{\Gamma(n + \frac{1}{2})}{2n!} = \frac{\sqrt{\pi}}{2^n} \frac{1}{2n(2n-2)(2n-4) \cdots 4 \cdot 2} = \frac{\sqrt{\pi}}{4^n n!}$$

so that (31) takes the form

$$i(x, t) = \frac{V_m}{\sqrt{r/c}} \frac{1}{\sqrt{\pi t}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{a}{4t}\right)^n \quad (33)$$

which we recognize as the conventional series expansion of (27), absolutely convergent, and particularly useful for small values of distance x because $a = rcx^2$, and for large values of time t .

Before discussing the physical aspects of the current solution, let us also find the voltage distribution along the cable because we can use most of the above developments. We note from the Laplace transform (10) of the voltage that because of the extra factor $1/p$ the inverse transform can be written as the integral

$$v(x, t) = V_m \int_{t=0+}^t (\mathfrak{L}^{-1} e^{-\sqrt{ap}}) dt \quad \text{Re } \sqrt{p} > 0 \quad (34)$$

Therefore, we need to find the inverse Laplace transform indicated in the parentheses, namely

$$\frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^{-\sqrt{ap}} e^{pt} dp \quad (34a)$$

Except for the factor $1/\sqrt{p}$ this is identical with (22) and thus permits the same demonstration that the value of the integral is identical with the contributions along $E''G''$ and $G'E'$. Using (23) we get integrals similar to (24) which now contract into

$$\frac{-1}{2\pi j} \int_{\infty}^0 e^{j\sqrt{a\mu}} e^{-\mu t} d\mu + \frac{-1}{2\pi j} \int_0^{\infty} e^{-j\sqrt{a\mu}} e^{-\mu t} d\mu = \frac{1}{\pi} \int_0^{\infty} \sin \sqrt{a\mu} e^{-\mu t} d\mu \quad (35)$$

* Jahnke-Emde, *op. cit.*, p. 11.

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This is obviously again a direct Laplace integral in the real variable μ (instead of t) with the parameter t (instead of p). We find this in the Campbell-Foster tables,^{E1} as pair 653.4, namely

$$\mathcal{L} \left(\sqrt{\frac{c}{\pi}} \sin 2 \sqrt{\frac{t}{c}} \right) = \frac{1}{p^{3/2}} \exp \left(-\frac{1}{cp} \right) \quad (36)$$

With the appropriate transposition, replacing in this result p by t and c by $4/a$ and taking $\sqrt{c/\pi}$ on the right-hand side, we obtain

$$\int_0^\infty \sin \sqrt{a\mu} e^{-\mu t} d\mu = \frac{1}{2} \sqrt{\pi a} t^{-3/2} \exp \left(-\frac{a}{4t} \right) \quad (37)$$

so that the integrand in (34) has the final form

$$\mathcal{L}^{-1} e^{-\sqrt{ap}} = \frac{1}{\sqrt{\pi t}} \sqrt{\frac{a}{4t}} \exp \left(-\frac{a}{4t} \right) \quad (38)$$

We should, of course, demonstrate a direct evaluation of the integral (34a). This is rather simple if we follow the method demonstrated for the current. We can use the Taylor series expansion (28) directly, drop at once the integer powers of p including $p^0 = 1$ because their contributions cancel over $E''G''$ and $G'E'$. The terms having attached $p^{1/2}$ have identical typical forms as in (29), so that (30) again applies. Collecting for the sum of half power terms in (28), we have

$$\mathcal{L}^{-1} e^{-\sqrt{ap}} = - \sum_{n=1}^{\infty} \frac{a^{n-1/2}}{(2n-1)!} \left(\frac{(-1)^n}{\pi} \frac{\Gamma(n + \frac{1}{2})}{t^{n+1/2}} \right) \quad (39)$$

where the part in large parenthesis is identical with (31). If we use again the explicit form (32) for the gamma function and combine it with $(2n-1)!$, (39) can be reduced to

$$- \frac{1}{\sqrt{\pi t}} \frac{2}{\sqrt{a}} \sum_{n=1}^{\infty} (-1)^n \left(\frac{a}{4t} \right)^n \frac{1}{(n-1)!}$$

To obtain the series in a form that we can identify it as the expansion of the exponential in (38), we replace in the sum n by $(n+1)$ so that it starts at $n=0$

$$\sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{a}{4t} \right)^{n+1} \frac{1}{n!} = - \frac{a}{4t} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{a}{4t} \right)^n \quad (40)$$

and take outside, as shown, the common factor $a/(4t)$. The final form for (39) is now

$$\mathfrak{L}^{-1}e^{-\sqrt{ap}} = \frac{1}{\sqrt{\pi}t} \sqrt{\frac{a}{4t}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{a}{4t}\right)^n \quad (41)$$

which is actually identical with (38).

The integral expression (34) for the voltage simplifies if we introduce the change in variable

$$\frac{4t}{a} = \tau = \frac{1}{y^2}, \quad \frac{4}{a} dt = d\tau = -\frac{dy}{y^3} \quad (42)$$

both in (34) and (38) which leads to

$$v(x,t) = -V_m \frac{2}{\sqrt{\pi}} \int_{y=\infty}^{1/\sqrt{\tau}} e^{-y^2} dy$$

Fortunately, this type of integral has been tabulated as *error integral**

$$\operatorname{erf}(y) = E_2(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-y^2} dy, \quad \operatorname{erf}(\infty) = 1 \quad (43)$$

$$\operatorname{erfc}(y) = \frac{2}{\sqrt{\pi}} \int_y^{\infty} e^{-y^2} dy = 1 - \operatorname{erf}(y) \quad (44)$$

Comparison shows that by interchange of the limits and corresponding change of sign, our integral is identical with (44), so that

$$v(x,t) = V_m \frac{2}{\sqrt{\pi}} \int_{1/\sqrt{\tau}}^{\infty} e^{-y^2} dy = V_m \operatorname{erfc}\left(\frac{1}{\sqrt{\tau}}\right) = V_m \operatorname{erfc}\left(\frac{x}{2} \sqrt{\frac{rc}{t}}\right) \quad (45)$$

This result can be graphically represented by a single curve as in Fig. 7.4 with $\tau = (4t)/(rcx^2)$ as the variable. Since x^2 and t occur in reciprocal combination we can conclude at once that the voltage build-up in time is delayed inversely with the square of the distance, the famous *KR-law* of Lord Kelvin† which predicted the severe limitation upon speed of telegraph transmission over the transatlantic cable.

* Jahnke-Emde, *op. cit.*, pp. 23-31; McLachlan,^{D13} pp. 78-83; and almost any book on advanced calculus.

† W. Thomson (Lord Kelvin), "On the Theory of the Electric Telegraph," *Proc. Royal Soc.*, **7**, 382 (1856). Also in *Phil. Mag.* (4), ii, 146-160 (1856). See J. R. Carson, *Electric Circuit Theory and Operational Calculus*, chapter VI, McGraw-Hill, New York, 1926.

If we take $rx = R$, $cx = C$ as the total resistance and capacitance (designated K by Lord Kelvin), respectively, between transmitter and receiver, then the reciprocal of the product can be considered as a measure of the permissible speed of telegraphy whatever the actual length of the cable. We will return to this later again.

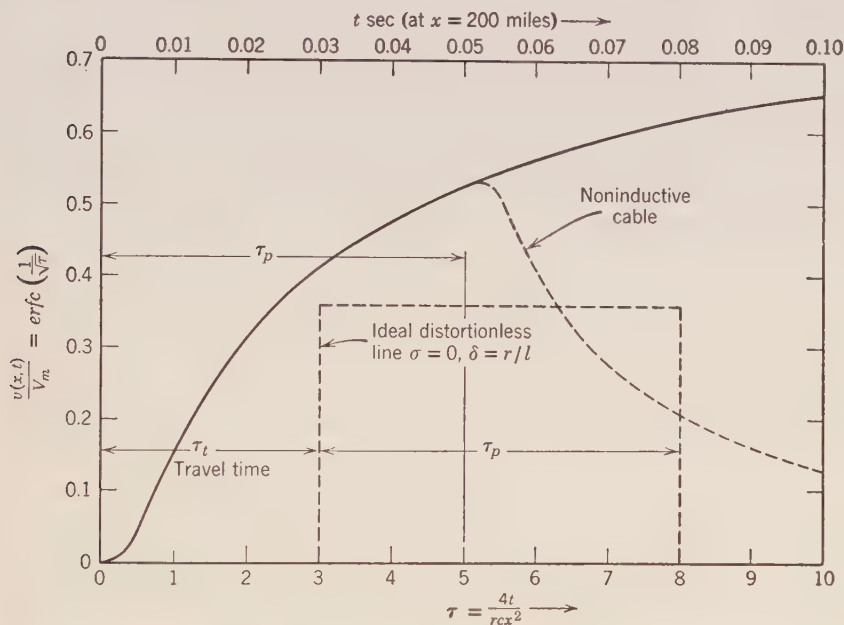


Fig. 7.4. Voltage pulse shape on infinite noninductive cable. Comparison with ideal distortionless line.

In order to appreciate the magnitudes involved, we might assume for a nonloaded deep-sea cable the approximate values $r = 2.5$ ohms/mile, $c = 0.40$ μ farad/mile so that

$$\tau = \frac{4t}{x^2} 10^6 = 4 \frac{100t}{(x/100)^2}$$

where we take time t in seconds and measure distance in (nautical) miles. At a distance $x = 200$ miles we have $\tau = 100t$, or we can interpret the horizontal scale directly in hundredths of seconds as indicated on the upper scale in Fig. 7.4; it will take 0.1 sec until the voltage reaches $\frac{2}{3}V_m$. If we apply a single square pulse of voltage of duration $t_p = 0.05$ sec or $\tau_p = 5$ at the sending end at $t = 0$, its oscillogram is given by the solid curve up to $\tau = 5$ and continued along the dashed curve in accordance with the superposition principle. At a distance

$x = 200$ miles we need only to use the upper scale to get the absolute time values. Let us now take the same cable conductor but wrap Permalloy tape about it so as to load it uniformly, making the inductance approximately $l = 0.06$ henry/mile; let us also assume the appropriate leakance value $g = (r/l)c$ to obtain a distortionless line as defined by (6.34). The velocity of propagation is

$$u = \frac{1}{\sqrt{lc}} = (0.06 \times 0.4 \times 10^{-6})^{-1/2} = 6500 \text{ miles/sec}$$

giving a travel time to $x = 200$ miles of $t = \frac{200}{6500} = 0.03$ sec or $\tau = 3$.

The pulse will thus appear in the oscillogram of the distortionless cable as an exact replica of the applied rectangular pulse, starting at $\tau = 3$, lasting an interval $\tau = 5$, and having the amplitude

$$e^{-\frac{\delta x}{u}} = e^{-\frac{r x}{l u}} = \exp\left(-\frac{2}{0.06} \frac{200}{6500}\right) = 0.36$$

as shown in Fig. 7.4. It is at once apparent that the noninductive cable does not show any real propagation effect; the voltage builds up along the entire infinitely long cable as if it had infinite velocity of propagation, though the voltage values for $\tau < 0.10$ are very small. The principal disadvantage of the noninductance cable is the smearing out of the pulse shape over a time interval several times as long as the pulse duration so that successive pulses will flow into each other unless distinctly spaced. Thus, a rather definite limit of resolution will exist which defines the speed of telegraph transmission. Loading will tend to reduce this distortion and, in the ideal case of the true distortionless line, permit any speed of telegraph transmission.

Actually, it is the current that is utilized in the activation of relays at the receiving end. We found the solution for the current in (27), which we might write with $\tau = (4t)/(rcx^2)$

$$i(x, t) = \frac{2}{\sqrt{\pi}} \frac{V_m}{xr} \left(\frac{1}{\sqrt{\tau}} e^{-1/\tau} \right) = \frac{2}{\sqrt{\pi}} \frac{V_m}{xr} \phi(\tau) \quad (46)$$

This form is most advantageous for a fixed distance x . We can, indeed, concentrate upon the function $\phi(\tau)$ which gives the essential features of the current as a function of time and is shown in Fig. 7.5. If we consider telegraph signals as rectangular voltage pulses of duration T_p or, expressed numerically, $\tau_p = [4/(rcx^2)] T_p$, then the current responses will be as shown in curves a, b, c of Fig. 7.5 in accordance with the law of superposition. To express this analytically, the

applied voltage pulse is defined as the difference of two step voltages

$$v(t) = V_m[S_{-1}(t) - S_{-1}(t - T_p)]$$

and the current response is correspondingly

$$i(x,t) = \frac{2}{\sqrt{\pi}} \frac{V_m}{xr} [\phi(\tau) - \phi(\tau - \tau_p)S_{-1}(t - T_p)] \tag{47}$$

where the unit step factor indicates that the second term is zero for $t < T_p$. Once the pulse duration $\tau_p \leq \frac{1}{2}$, the maximum value of the current response will be very nearly proportional to τ_p so that the

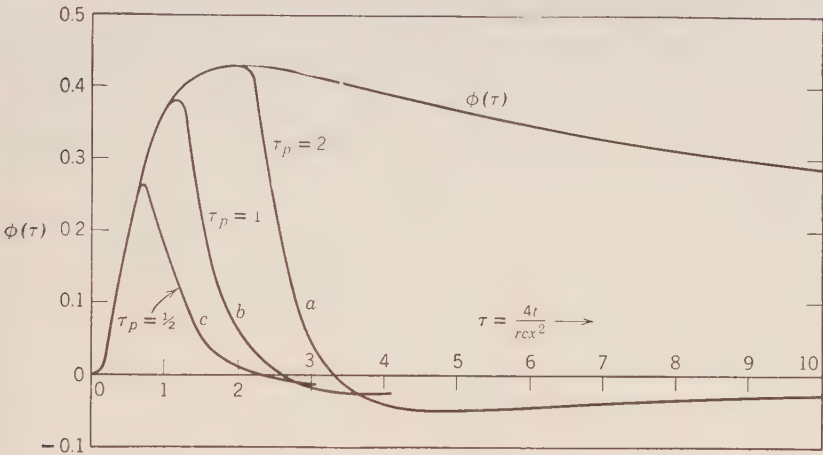


Fig. 7.5. Current response of infinite noninductive cable. $\phi(\tau)$, response to step voltage; a, b, c responses to voltage pulses of durations $\tau_p = 2, 1, \frac{1}{2}$, respectively.

cable transforms the code of voltage pulses of varying duration into current pulses of proportionally varying amplitudes! This can be demonstrated by developing the second term in (47) into a Taylor series

$$\phi(\tau - \tau_p) = \phi(\tau) - \tau_p \frac{d\phi(\tau)}{d\tau} + \frac{\tau_p^2}{2} \frac{d^2\phi(\tau)}{d\tau^2} + \dots$$

and introducing this into (47) for $\tau > \tau_p$ but close to τ_p

$$i(x,t)_{t>T_p} = \frac{2}{\sqrt{\pi}} \frac{V_m}{xr} \tau_p \frac{d\phi(\tau)}{d\tau} \tag{48}$$

Because the slope of $\phi(\tau)$ is nearly constant in the range $0.2 < \tau < 0.75$, (48) demonstrates that the maximum, which occurs for values of τ slightly larger than τ_p , is proportional to τ_p .

7.3 Power Series Developments of Solutions

We have already observed that the inverse Laplace transforms of diffusion problems can lead to difficult integrals in the complex plane and that we might have to resort to series expansions. It is therefore important to discuss a few simpler criteria for establishing the validity of the results obtained by various types of power series expansions either in the real t -domain or in the transform domain of the complex variable p . To be general, let us use for these discussions the conventional symbol z for the complex variable and specify p or t only in the actual applications.

If we take the simple example of the binomial expansion

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \quad (49)$$

we realize that it *converges* for $|z| < 1$ *uniformly*, i.e., for every value z within the unit circle or the *radius of convergence* $|z| = \rho < 1$. We must avoid $z = 1$ because it is a pole of the function $(1-z)^{-1}$. Any other expansion in the regular region of this function is likewise valid up to this singularity $z = 1$ (see Appendix 5). For values $|z| > 1$, (49) is clearly *divergent* and cannot be used at all. However, we can rearrange and obtain by direct expansion

$$\frac{-1}{z-1} = -\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) \quad (50)$$

which is now uniformly convergent for $|z| > 1$ and diverges for $|z| \leq 1$. In the real domain, (49) is the geometric series that is often taken as a standard of convergence with which a newly formed series might be compared.

In general, we might expand any given function in either ascending or descending power series whereby each case requires individual examination to determine convergence or divergence. Suppose we find upon expansion

$$f(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \cdots + a_nz^n + \cdots \quad (51)$$

and we know the general law of generating the coefficients a_n as a function of n . If we designate the sum of the terms including a_nz^n as $S_n(z)$, and the sum of the succeeding infinite number of terms as the remainder $R_n(z)$, then

$$f(z) = S_n(z) + R_n(z) \quad (51a)$$

will be represented by a *convergent* series if

$$\lim_{n \rightarrow \infty} R_n(z) \rightarrow 0$$

This is a necessary *and* sufficient condition but is usually difficult to demonstrate. It is easier to show that the sum of the absolute values of the terms converges, i.e., that

$$|a_0| + |a_1z| + |a_2z^2| + \cdots + |a_nz^n| + \cdots \quad (52)$$

exists and that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n z^n|} = \lim_{n \rightarrow \infty} |a_n|^{1/n} |z| < 1 \quad (53)$$

This is a *sufficient* condition of *absolute convergence*, and is actually equivalent to the condition $|z| < 1$ in (49). Just as we called $|z| = \rho < 1$ the radius of convergence, so we can define here the radius of absolute convergence (Cauchy-Hadamard criterion)

$$\lim_{\text{conv}} |z| < \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}} \quad (54)$$

Obviously, if (52) converges, then certainly (51) will converge *uniformly* over at least the same range of the variable.

An equivalent test for absolute convergence is the *ratio test*. From (49) we see that the ratio of the absolute values of two successive terms leads to the identical form $|z| < 1$ as the condition for absolute convergence. Applying this to (52) we have

$$\lim_{n \rightarrow \infty} \left| \frac{a_n z^n}{a_{n-1} z^{n-1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n-1}} \right| |z| < 1 \quad (55)$$

which we can also formulate in terms of radius of convergence

$$\lim_{\text{conv}} |z| < \frac{1}{\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n-1}} \right|} \quad (56)$$

Any power series which is absolutely convergent is also uniformly convergent within the radius of convergence; this assures us that we can integrate and differentiate term by term within this range of the variable and that the radius of absolute convergence of the integral and derivative will be the same as for the original series. The latter follows most readily from (55) and (56). Differentiating (51) term by term brings the factors n , $(n - 1)$ and conversely, integrating (51)

term by term, the factors $1/(n+1)$, $1/n$ to a_n , a_{n-1} , respectively. Since

$$\lim_{n \rightarrow \infty} \frac{n}{n-1} = \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1 \quad (57)$$

the ratio test on the derivative or integral series gives the same result as (56).

A series which does not satisfy (53) or (55) is certainly not absolutely convergent, yet it might converge for certain values of the variable. If we can demonstrate its convergence by some means, we would call it *conditionally convergent*.

Series which are not convergent are called *divergent* but they still can be useful. One class of divergent series has attained considerable practical importance because it can be used to approximate functions for certain ranges of the variable without being an exact representation such as the convergent series is. Because the first few terms up to $n = N$ converge and only the terms for $n > N$ diverge, these series are also designated as *semiconvergent* or *semidivergent*. The general theory of these series is rather involved; we shall be concerned here only with the *asymptotic* series which is a special type of semiconvergent development.

Suppose we find that (51) is a nonvergent expansion, which means that the coefficients a_n increase indefinitely with the order number n . Such a series might be useful but only for small values of z . We might take again as in (51a) the sum $S_n(z)$ of the first terms including order number n and call the sum of all further terms the remainder $R_n(z)$ as in (51a). If we can demonstrate that

$$\lim_{z \rightarrow 0} R_n(z) \rightarrow 0 \quad (58)$$

then we call $S_n(z)$ the *asymptotic representation* of the function $f(z)$ from (51). We observe that this is a rather different requirement than for the convergent series (51a)! In essence it means that for a fixed order number n we get better and better approximation of $f(z)$ by the polynomial $S_n(z)$ as $z \rightarrow 0$. Indeed, we can find the coefficients of the asymptotic expansion, if the latter exists, by successively forming

$$\begin{aligned} \lim_{z \rightarrow 0} f(z) &\rightarrow a_0 \\ \lim_{z \rightarrow 0} [f(z) - a_0]/z &\rightarrow a_1 \\ \lim_{z \rightarrow 0} [f(z) - a_0 - a_1 z]/z^2 &\rightarrow a_2 \end{aligned} \quad (59)$$

etc. Asymptotic series can be integrated generally, but not differentiated without careful demonstration of the validity of the result. Extensive discussion of the asymptotic representation of functions are found in Bush,* chapter XII, Carson,† chapter V, Doetsch,^{D5} part III, and Van der Pol-Bremmer,^{D15} chapter VII.

All the discussion can be applied with the appropriate modifications to power series in descending powers of z . Thus, (50) and (31) were examples of convergent series developments. The general form

$$f(z) = b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots + \frac{b_n}{z^n} \cdots \quad (60)$$

could be an asymptotic representation for $z \rightarrow \infty$ if we were able to demonstrate

$$\lim_{z \rightarrow \infty} R_n(z) \rightarrow 0 \quad (61)$$

for a fixed value of n .

In the case of power series in a real variable, say t , we can add another valuable observation. If the convergent series is *alternating*, i.e., successive coefficients have alternating signs, we shall write it with all a_n positive as

$$g(t) = a_0 - a_1 t + a_2 t^2 - a_3 t^3 + \cdots \pm a_n t^n \pm \cdots \quad (62)$$

where the sign of the n 'th term depends on whether n is even or odd.

The remainder $R_n(t)$ can then be written in two ways, using $\mu_n = a_n t^n$

$$\begin{aligned} R_n(t) &= \mp [(\mu_{n+1} - \mu_{n+2}) + (\mu_{n+3} - \mu_{n+4}) + \cdots] \\ &= \mp [\mu_{n+1} - (\mu_{n+2} - \mu_{n+3}) - (\mu_{n+4} - \mu_{n+5}) - \cdots] \end{aligned}$$

If we have chosen n large enough so that the terms continuously decrease

$$\mu_{n+1} > \mu_{n+2} > \mu_{n+3} > \cdots$$

then we find the bracket in the first line to be a sum of all positive terms so its value is larger than zero, whereas the bracket in the second line gives a value definitely less than μ_{n+1} . Breaking off in (62) with $\mu_n = \pm a_n t^n$ as the last term means an error less than the succeeding term $\mu_{n+1} = a_{n+1} t^{n+1}$ and of opposite sign! By judicious interpolation we can make this error very small. Alternating convergent series therefore permit a direct and simple evaluation of the maximum error in the value of the function by using $S_n(t)$ as representation.

* V. Bush, *Operational Circuit Analysis*, Wiley, New York, 1929.

† J. R. Carson, *op. cit.*, chapter V.

The same reasoning can be applied to alternating asymptotic series of a real variable if we select $S_n(t)$ to include all decreasing terms and $R_n(t)$ to include the increasing terms. Actually, one should break off the asymptotic series with the smallest term in order to assure the smallest possible error.

7.4 Asymptotic Laplace Transform Relations

Unfortunately, it is not possible to formulate very general relations between series expansions in the time domain and in the Laplace transform domain even though considerable study has been given to this subject, mostly based upon theorems originally developed by Abel and Tauber; see Doetsch,^{D5} part III, Van der Pol-Bremmer,^{D15} chapter VII, Widder,^{D24} and Wiener.^{D25}

We can get a reasonably sound orientation from the following considerations. Suppose we have found the Laplace transform $F(p)$ as the transform solution of a physical problem so that we already have defined its characteristics as a function of the complex variable p , i.e., we have selected the analytic branch constituting the physical solution and know all its singularities and their locations. Good examples are the solutions (13), (22), and (34a) for which we fortunately could find "closed form" inverse transforms; let us designate these as $f(t)$.

According to the nature of $f(t)$, we might be able to expand it near the origin for small values of t into a power series

$$f(t) = f(0^+) + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \quad (63)$$

which might be either convergent or asymptotic depending upon the values of the coefficients. The Laplace transform of (63) is term by term from Table 1.3

$$F(p) = \frac{f(0^+)}{p} + \frac{a_1}{p^2} + 2! \frac{a_2}{p^3} + 3! \frac{a_3}{p^4} + \cdots \quad (64)$$

and this will, more likely than not, be the beginning of an asymptotic series in negative powers of p valid for $p \rightarrow \infty$. We take from (63) and (64) at once the relations

$$\begin{aligned} \lim_{p \rightarrow \infty} pF(p) &\rightarrow f(0^+) \\ \lim_{p \rightarrow \infty} p[pF(p) - f(0^+)] &\rightarrow a_1 = \left(\frac{d}{dt} f(t) \right)_{t=0} \end{aligned} \quad (65)$$

which we had deduced as the initial value relation 1a in Table 1.4. Inverting the process, we might state: If we can expand the Laplace

TABLE 7.1
SHORT TABLE OF SIMPLE IRRATIONAL LAPLACE TRANSFORMS
(DIFFUSION PROBLEMS)

No.	Time Domain	Transform Domain	Relation in Text (Section 7)
1	$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{p}}$	21
2	$-\frac{1}{2\sqrt{\pi} t^{3/2}}$	\sqrt{p}	
3	$\frac{(-1)^n}{\pi} \frac{\Gamma(n + \frac{1}{2})}{t^{n+1/2}}$ $= \frac{(-1)^n}{\sqrt{\pi t}} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2t)^n}$	$p^{n-1/2}$	29, 31
4	$\frac{(-1)^n}{\pi} \Gamma(-n + \frac{1}{2}) t^{n-1/2}$ $= \frac{1}{\sqrt{\pi t}} \frac{(2t)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$	$p^{-n-1/2}$	
5	$\frac{1}{\sqrt{\gamma}} e^{\gamma t} \text{erf}(\sqrt{\gamma t})$	$\frac{1}{p-\gamma} \frac{1}{\sqrt{p}}$	75
6	$e^{\gamma t} \text{erfc}(\sqrt{\gamma t})$	$\frac{1}{\sqrt{p}(\sqrt{p} + \sqrt{\gamma})}$	73a, 77
7	$e^{-\delta t} \frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{p+\delta}}$	112
8	$\frac{1}{\sqrt{\delta}} \text{erf}\sqrt{\delta t}$	$\frac{1}{p\sqrt{p+\delta}}$	113
11	$\frac{1}{\sqrt{\pi} t} \sqrt{\frac{a}{4t}} \exp\left(-\frac{a}{4t}\right)$	$e^{-\sqrt{ap}}$	38
12	$\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{a}{4t}\right)$	$\frac{1}{\sqrt{p}} e^{-\sqrt{ap}}$	26
13	$\frac{1}{\sqrt{\pi t}} \cos 2\sqrt{\frac{t}{a}}$	$\frac{1}{\sqrt{p}} \exp\left(-\frac{1}{ap}\right)$	25
14	$\sqrt{\frac{a}{\pi}} \sin 2\sqrt{\frac{t}{a}}$	$\frac{1}{p^{3/2}} \exp\left(-\frac{1}{ap}\right)$	36
15	$\text{erfc}\left(\sqrt{\frac{a}{4t}}\right)$	$\frac{1}{p} e^{-\sqrt{ap}}$	10, 45

transform solution $F(p)$ into a descending power series in p , either convergent or more likely asymptotic, then its inverse Laplace transform, term by term, will constitute a convergent, occasionally an asymptotic, power series in t valid for small values of t . The presence of a constant term in (64) indicates a unit impulse at $t = 0$. We must obviously be sure that p is chosen large enough to have the region of expansion (64) completely outside all isolated singularities of the finite p -plane. In fact, if we substitute $p = 1/q$, then (64) can be considered as the expansion of $F(1/q)$ about $q = 0$ valid to the nearest singularity.

As examples (13), (22), and (34a) have shown, transmission-line problems frequently involve functions of the square root of p . If we introduce temporarily $\sqrt{p} = W$, and expand $F(p)$ in negative powers of W , we can segregate the even powers as integer powers of p like (64) from the odd powers of W which constitute integer powers of p multiplied by \sqrt{p} , so that

$$F(p) = \left(\frac{A_1}{p} + \frac{A_2}{p^2} + \frac{A_3}{p^3} + \cdots \right) + \left(\frac{B_1}{p} + \frac{B_2}{p^2} + \frac{B_3}{p^3} + \cdots \right) \sqrt{p} \quad (66)$$

As long as the individual series are both either convergent or asymptotic, we can still transpose them term by term, using for the terms of the half power series the applicable pairs from Table 7.1. For asymptotic series, we need to demonstrate (61), to be sure. The resultant time series will have the form

$$f(t) = \left(A_1 + A_2 \frac{t}{1!} + A_3 \frac{t^2}{2!} + \cdots \right) + \left(B_1 + B_2 \frac{2t}{1} + B_3 \frac{(2t)^2}{1 \cdot 3} + \cdots \right) \frac{1}{\sqrt{\pi t}} \quad (67)$$

and is usually convergent. It is important to check that this time series is not divergent; should it turn out to be divergent, then the series development (66) was improper.

As one simple example we might choose

$$F(p) = \frac{\omega}{p^2 + \omega^2} = \frac{\omega}{p^2} \left[1 - \left(\frac{\omega}{p} \right)^2 + \left(\frac{\omega}{p} \right)^4 - \left(\frac{\omega}{p} \right)^6 + \cdots \right]$$

This series converges absolutely for *large* values of p ; the inverse Laplace transform term by term gives the convergent series

$$f(t) = \omega \left(t - \omega^2 \frac{t^3}{3!} + \omega^4 \frac{t^5}{5!} - \cdots \right) = \sin \omega t$$

As another example we might verify (25). Expanding the exponential we obtain

$$\begin{aligned} F(p) &= \frac{1}{\sqrt{p}} \exp\left(-\frac{1}{ap}\right) \\ &= \frac{1}{\sqrt{p}} \left(1 - \frac{1}{ap} + \frac{1}{2!} \frac{1}{(ap)^2} - \frac{1}{3!} \frac{1}{(ap)^3} + \cdots\right) \end{aligned}$$

which is an absolutely convergent series for any value of $p \neq 0$. This is reasonable because $F(p)$ has no singularities in the finite p -plane except the branch point at $p = 0$. The inverse transform term by term gives with pair 4, Table 7.1

$$f(t) = \frac{1}{\sqrt{\pi t}} \left(1 - \frac{2t}{a} + \frac{1}{2!} \frac{(2t)^2}{a^2 1 \cdot 3} - \frac{1}{3!} \frac{(2t)^3}{a^3 1 \cdot 3 \cdot 5} + \cdots\right)$$

If we multiply numerator and denominator of each term by 2^n , where n indicates the power of $2t$, we can contract the terms into a simpler form, e.g., the third term

$$\frac{1}{2!} \left(\frac{2t}{a}\right)^2 \frac{1}{1 \cdot 3} = \left(\frac{4t}{a}\right)^2 \frac{1}{4!}$$

Thus the series becomes

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{\pi t}} \left[1 - \frac{1}{2!} \frac{4t}{a} + \frac{1}{4!} \left(\frac{4t}{a}\right)^2 - \frac{1}{6!} \left(\frac{4t}{a}\right)^3 + \cdots\right] \\ &= \frac{1}{\sqrt{\pi t}} \cos 2\sqrt{\frac{t}{a}} \end{aligned}$$

which holds for all values of t and proves relation (25).

In analogy to (63), we would like to be able to expand $f(t)$ for very large values of t into a power series

$$f(t) = f(\infty) + \frac{b_1}{t} + \frac{b_2}{t^2} + \frac{b_3}{t^3} + \cdots \quad (68)$$

which we would expect to be of the asymptotic type but which might also be on occasions a convergent series, as, e.g.

$$\exp\left(-\frac{a}{4t}\right) = 1 - \frac{a}{4t} + \frac{1}{2!} \left(\frac{a}{4t}\right)^2 - \frac{1}{3!} \left(\frac{a}{4t}\right)^3 + \cdots$$

But we would be at a loss to construct for it the Laplace transform series! If we were to proceed in a straightforward manner we find

$$\mathfrak{L} \frac{b_n}{t^n} \rightarrow \int_0^\infty \frac{b_n}{t^n} e^{-pt} dt = b_n p^{n-1} \int_0^\infty e^{-y} y^{-n} dy$$

where we introduced $y = pt$. The integral can be identified from (19) as the gamma function, but $\Gamma(-n) = \infty$ for any integer value* of n including $n = 0$. This means that the Laplace transforms of the terms in (68) do not exist individually except the first which gives $(1/p)f(\infty)$; this is the term we have entered in line 1b of Table 1.4, expressing the relation for the ultimate value of the time function. Extension of pair 3, Table 1.3, to negative values of n is thus impossible and, indeed, we know that the positive powers in p are the Laplace transforms of the highly discontinuous impulse functions.

We see, therefore, that the asymptotic approximation of $f(t)$ for large values of t is much more difficult than that for $t \rightarrow 0$. We need only remember that all the rational functions in p (fractions of two polynomials) have as inverse Laplace transforms a sum of exponentials $e^{p_\alpha t}$ which cannot be represented by asymptotic power series of the type (68) valid for large values of t . If the root values p_α are negative real, we might disregard the exponentials as $t \rightarrow \infty$; if, however, there are conjugate complex or, even worse, conjugate imaginary root values, then we have periodic functions, damped or undamped, which need to be enumerated separately to characterize the behavior of $f(t)$ for large t .

On the other hand, we have already seen how $e^{-\sqrt{ap}}$ in (28) could be expanded into a convergent positive power series of \sqrt{p} , having as inverse Laplace transform the also convergent time series (39) or (41) valid for all values of t but particularly useful for large t . Since $e^{-\sqrt{ap}}$ has no singularities in the finite p -plane except the branch point (but no pole) at $p = 0$, the expansion about the origin $p = 0$ is valid in the entire finite p -plane, explaining the regularity of both p and t series. In finding the inverse Laplace transform we also noted that the even powers of \sqrt{ap} leading to positive integer powers in p do not contribute to the time function as shown in detail following (29); we can therefore disregard these positive integer powers of p in any such series development. In a more general case, the series in positive powers of \sqrt{p} might have a definite radius of convergence about $p = 0$ because of poles at a finite distance from the origin. In such case we have to examine these poles, and if they are conjugate complex and lie within the branch of \sqrt{p} for which the expansion is valid then we

* See Jahnke-Emde, *op. cit.*, p. 11 and Fig. 7 on p. 15.

have to add their contribution to $f(t)$ separately because the series will not expose them.

To summarize now: To portray the asymptotic behavior of a function $f(t)$ for large values of time t directly from the known Laplace transform $F(p)$, it is advisable that we separate as far as possible the rational and the irrational parts

$$F(p) = \frac{N(p)}{D(p)} + H(\sqrt{p}) \tag{69}$$

To the rational part we apply

$$f(\infty) = \lim_{p \rightarrow 0} \left(p \frac{N(p)}{D(p)} \right) \tag{70}$$

giving the final stationary value of $f(t)$ if such a value exists. To have $f(\infty) \neq 0$ requires $D(p) = pD_1(p)$, i.e., a separate factor p in the denominator, as well as a nonvanishing constant term in $N(p)$. Next we check upon the roots of $D(p)$ and write the residues for all conjugate complex root pairs which will be the damped oscillations of the system. Then we turn to the irrational part and examine its poles within the branch of the solution, writing out explicitly the residues at the conjugate complex pairs of poles which expose any other damped oscillations of the system. Finally, we expand $H(\sqrt{p})$, if feasible, into ascending powers of \sqrt{p} and order

$$H(\sqrt{p}) = \frac{1}{\sqrt{p}} (B_0 + B_1p + B_2p^2 + B_3p^3 + \cdots) \\ + (A_1p + A_2p^2 + A_3p^3 + \cdots) \tag{71}$$

If the series are convergent or at most asymptotic, then we drop the second line and convert the first line, term by term, into the inverse Laplace transform by pair 3 of Table 7.1, so that

$$h(t) = \frac{1}{\sqrt{\pi t}} \left(B_0 - B_1 \frac{1}{2t} + B_2 \frac{1 \cdot 3}{(2t)^3} - B_3 \frac{1 \cdot 3 \cdot 5}{(2t)^5} + \cdots \right) \tag{72}$$

This is generally an asymptotic series in t valid for large values of t only.

7.5 Input Voltage on Cable Charged Over a Lumped Circuit Element

The various asymptotic series developments discussed in the preceding section can be illustrated best by solving some relatively simple problems. We will first find the input voltage to the infinitely long

noninductive cable if a step voltage is applied to it in series with a lumped capacitance C as shown in Fig. 7.6a. The equivalent circuit

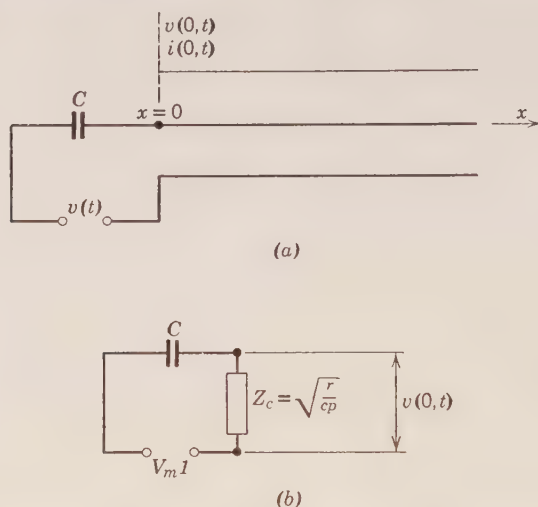


Fig. 7.6. Step voltage applied to noninductive cable over lumped capacitance C .
(a) Physical arrangement; (b) equivalent circuit.

is given in Fig. 7.6b; from it we find for the Laplace transform of the input voltage for the cable

$$V(0,p) = I(0,p)Z_c = \frac{Z_c}{Z_c + [1/(pC)]} \mathcal{L}v(t)$$

With Z_c from (5) and the Laplace transform V_m/p for the step voltage we have

$$V(0,p) = \frac{\sqrt{r/(cp)}}{\sqrt{r/(cp)} + 1/(pC)} \frac{V_m}{p} = \frac{V_m}{p + \sqrt{c/(rC^2)} \sqrt{p}} \quad (73)$$

If we define $\gamma = c/rC^2$, then we might write

$$V(0,p) = \frac{V_m}{\sqrt{p}(\sqrt{p} + \sqrt{\gamma})} \quad (73a)$$

We can find the inverse Laplace transform in closed form by multiplying the numerator and denominator by $(\sqrt{p} - \sqrt{\gamma})$, so that

$$\frac{1}{\sqrt{p}(\sqrt{p} + \sqrt{\gamma})} = \frac{\sqrt{p} - \sqrt{\gamma}}{\sqrt{p}(p - \gamma)} = \frac{1}{p - \gamma} - \frac{\sqrt{\gamma}}{p - \gamma} \frac{1}{\sqrt{p}} \quad (74)$$

The inverse transform of the first term is the exponential function from Table 1.3; that of the second term requires the application of the convolution integral from Table 1.5

$$F_1(p) = \frac{1}{p - \gamma}, \quad f_1(t) = e^{\gamma t} \text{ (Table 1.3, pair 2)}$$

$$F_2(p) = \frac{1}{\sqrt{p}}, \quad f_2(t) = \frac{1}{\sqrt{\pi t}} \text{ (Table 7.1, pair 1)}$$

so that

$$\mathcal{L}^{-1} \frac{1}{p - \gamma} \frac{1}{\sqrt{p}} = e^{\gamma t} \int_{t=0^+}^t e^{-\gamma \tau} \frac{1}{\sqrt{\pi \tau}} d\tau$$

If we introduce the variable $\gamma \tau = y^2$, $\gamma d\tau = 2y dy$, we can identify this integral as the error integral (43)

$$\mathcal{L}^{-1} \frac{1}{p - \gamma} \frac{1}{\sqrt{p}} = e^{\gamma t} \frac{2}{\sqrt{\pi \gamma}} \int_{y=0}^{\sqrt{\gamma t}} e^{-y^2} dy = \frac{1}{\sqrt{\gamma}} e^{\gamma t} \operatorname{erf}(\sqrt{\gamma t}) \quad (75)$$

The total inverse transform of (73a) is thus found with (74) and (75)

$$v(0, t) = V_m \left(e^{\gamma t} - \sqrt{\gamma} \frac{1}{\sqrt{\gamma}} e^{\gamma t} \operatorname{erf} \sqrt{\gamma t} \right) \quad (76)$$

If we observe (44), we can write this still simpler as

$$v_0(t) = V_m e^{\gamma t} \operatorname{erfc}(\sqrt{\gamma t}) \quad (77)$$

This function is shown in the solid line in Fig. 7.7. Since γ is a real positive quantity in (73a), we need not impose any restriction on the validity of our procedure. If we permit γ to be complex, then we must choose our path of integration definitely so that along it $\operatorname{Re} p > \operatorname{Re} \gamma$ as required by the definition of the inverse Laplace transform.

Although (77) has an extremely simple form and tables of both the exponential function and the error integral do exist, the actual computation for larger values of the variable γt is inconvenient because $e^{\gamma t}$ increases very fast and $\operatorname{erfc}(\sqrt{\gamma t})$ decreases at an even faster rate. We might therefore be interested in power series developments in order to find the inverse Laplace transform of (73a). Let us take the asymptotic series for $t \rightarrow \infty$ first. We have already separated rational and irrational parts as far as feasible in (74). The only pole is $p = \gamma$ or also $\sqrt{p} = -\sqrt{\gamma}$ as required by the first form of (74), occurring, therefore, in the branch of $H(\sqrt{p})$ in which we have no interest so that

we can disregard it henceforth. We see also

$$\lim_{p \rightarrow 0} \frac{p}{p - \gamma} \rightarrow 0$$

so that

$$\lim_{t \rightarrow \infty} v(x, t) \rightarrow 0 \quad (78)$$

We can now proceed to expand the second term in (74) into ascending powers of \sqrt{p} , which we do best by binominal expansion of

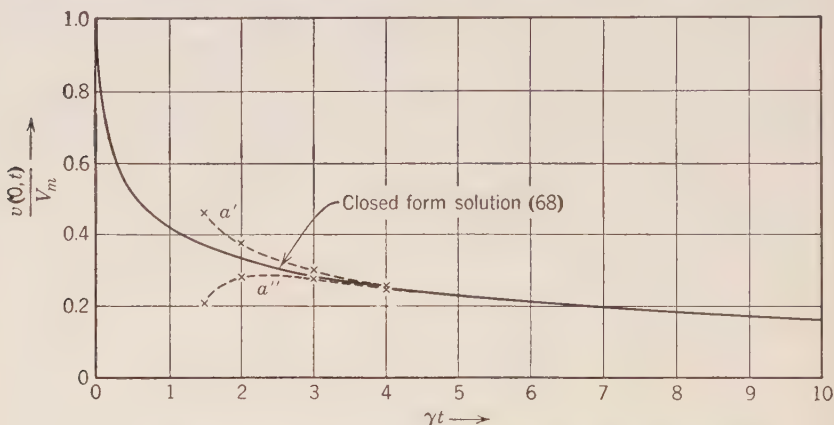


Fig. 7.7. Voltage variation at input end of noninductive cable charged over lumped capacitance C ; a' , a'' solution by asymptotic series including smallest and next to smallest terms, respectively.

$(p - \gamma)^{-1}$ so that

$$-\sqrt{\frac{\gamma}{p}} \frac{1}{p - \gamma} = \frac{1}{\sqrt{\gamma p}} \frac{1}{1 - (p/\gamma)} = \frac{1}{\sqrt{\gamma p}} \left[1 + \frac{p}{\gamma} + \left(\frac{p}{\gamma}\right)^2 + \dots \right] \quad (79)$$

This series converges only for small values of p but is directly in the form (71) without the integer powers in p , so that we can write the inverse time series at once as

$$v(0, t) = \frac{V_m}{\sqrt{\pi \gamma t}} \left(1 - \frac{1}{2\gamma t} + \frac{1 \cdot 3}{(2\gamma t)^2} - \frac{1 \cdot 3 \cdot 5}{(2\gamma t)^3} + \dots \right) \quad (80)$$

As $t \rightarrow \infty$ this demonstrates that the voltage approaches zero asymptotically like $1/\sqrt{t}$. Obviously, (80) is practical only for large values of γt ; it is a true asymptotic series expansion satisfying (61). We can readily demonstrate that it actually represents solution (77). This

is done best by returning to the defining integral (44) of the function $\operatorname{erfc}(y)$ and repeatedly integrating by parts in terms of the variable $\tau = y^2$, $d\tau = 2y dy$, as follows

$$\operatorname{erfc}(y) = \frac{2}{\sqrt{\pi}} \int_y^\infty e^{-y^2} dy = \frac{1}{\sqrt{\pi}} \int_{y^2}^\infty e^{-\tau} \frac{d\tau}{\sqrt{\tau}} = \frac{1}{\sqrt{\pi}} \int_{y^2}^\infty \frac{1}{\sqrt{\tau}} (e^{-\tau} d\tau)$$

Select $u = \frac{1}{\sqrt{\tau}}, \quad du = -\frac{d\tau}{2\tau^{3/2}}$

$$dv = e^{-\tau} d\tau, \quad v = -e^{-\tau}$$

so that

$$\sqrt{\pi} \operatorname{erfc}(y) = -\left. \frac{e^{-\tau}}{\sqrt{\tau}} \right|_{y^2}^\infty - \frac{1}{2} \int_{y^2}^\infty \frac{d\tau}{\tau^{3/2}} e^{-\tau} = \frac{e^{-y^2}}{y} - \frac{1}{2} \int_{y^2}^\infty \tau^{-3/2} (e^{-\tau} d\tau)$$

If we apply here the same type of integration by parts to the successive integral remainders, we end up with

$$\sqrt{\pi} \operatorname{erfc}(y) = e^{-y^2} \left(\frac{1}{y} - \frac{1}{2y^3} + \frac{1 \cdot 3}{2^2 y^5} - \cdots \right)$$

Introducing this expansion into (77) where $y = \sqrt{\gamma t}$, we readily verify that the result is identical with (80).

For the numerical computation, (80) is very much easier to use than the closed form (77). Because of the alternating sign the error is less than the first term omitted, so we can readily see that even for $\gamma t = 5$ the third term has only the value $+0.03$; its exclusion will mean a maximum error of -0.03 in $(1 - 0.1) = 0.9$ or -3.33% . We can reduce this error further by observing that the inclusion of the third term gives the bracket the value 0.93 with the fourth term defining now a maximum error of $+0.015$ or $+1.6\%$. Selecting instead two-thirds of the value of the third term leads to the good approximation in (80)

$$\left. \frac{v(0,t)}{V_m} \right|_{\gamma t=5} = \frac{1}{\sqrt{5\pi}} 0.91 = 0.229$$

whereas the exact value is 0.2285 . Curves a' and a'' in Fig. 7.7 give the values computed by (80) breaking off after the lowest term and the next following, respectively. As the curves clearly illustrate, the asymptotic series provides bounds between which the correct solution must lie. To emphasize this latter point better and more graphically, Fig. 7.8 shows the values of the asymptotic series for $\gamma t = 2$ as a function of the number of terms taken into account. The smallest term

is the third and we see that the series comes closest to the true value at $N = 3$. We can minimize the error by selecting a value between $N = 2$ and $N = 3$. There is no sense at all in using terms beyond the smallest because the sum of the series begins to fluctuate violently in keeping with its essentially divergent character.

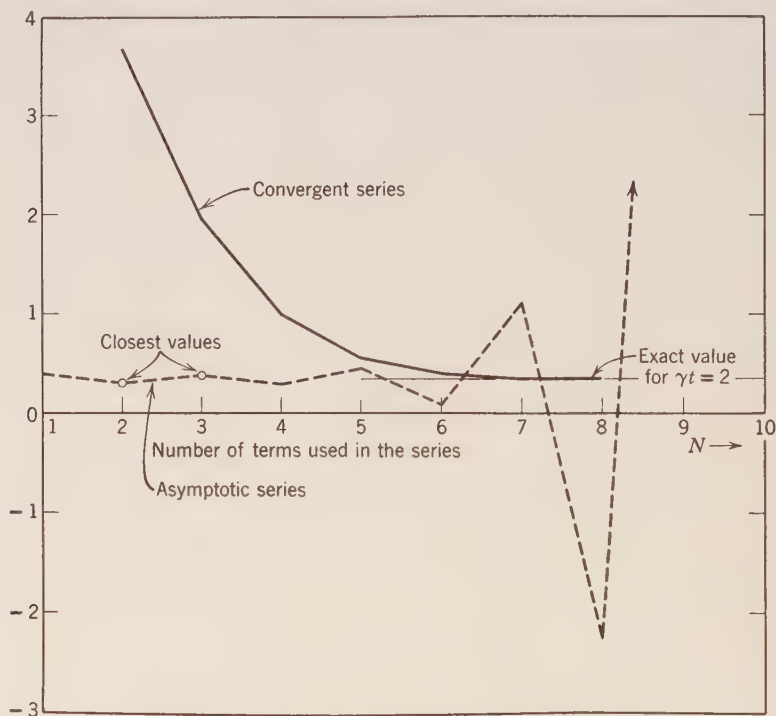


Fig. 7.8. Comparative values of the convergent series (84) and the asymptotic (80) as a function of the number of terms included for the value of the argument $\gamma t = 2$.

A suitable series for $t \rightarrow 0$ is obtained by expanding $(\sqrt{p} + \sqrt{\gamma})^{-1}$ in the first expression of (74) binominally into descending powers of \sqrt{p} , so that we get

$$\begin{aligned} \frac{1}{\sqrt{p}} \frac{1}{\sqrt{p} + \sqrt{\gamma}} &= \frac{1}{p} \frac{1}{1 + \sqrt{\gamma/p}} \\ &= \frac{1}{p} \left[1 - \left(\frac{\gamma}{p} \right)^{1/2} + \left(\frac{\gamma}{p} \right) - \left(\frac{\gamma}{p} \right)^{3/2} + \dots \right] \quad (81) \end{aligned}$$

We recognize that the series is absolutely convergent only for $|\gamma/p| < 1$. Separating the integer powers we have

$$\mathcal{L}^{-1}\left(\frac{1}{p} + \frac{\gamma}{p^2} + \frac{\gamma^2}{p^3} + \cdots\right) = 1 + \gamma t + \frac{(\gamma t)^2}{2} + \cdots \quad (82)$$

These are the first few terms of the exponential function $e^{\gamma t}$ which in turn is the inverse Laplace transform of the first term in the last expression (74) which we also could have used for the series expansion.

The terms with the factor $1/\sqrt{p}$ can be converted into the inverse time series by pair 4 of Table 7.1 so that

$$\begin{aligned} -\frac{1}{p}\sqrt{\frac{\gamma}{p}}\left(1 + \frac{\gamma}{p} + \frac{\gamma^2}{p^2} + \cdots\right) \\ = -\frac{1}{\sqrt{\pi\gamma t}}\left(\frac{2\gamma t}{1} + \frac{(2\gamma t)^2}{1 \cdot 3} + \frac{(2\gamma t)^3}{1 \cdot 3 \cdot 5} + \cdots\right) \end{aligned} \quad (83)$$

This constitutes a convergent series in time valid for small values of t in keeping with the limited convergence of (81). The final result is then for (73a)

$$v(0,t) = V_m \left[e^{+\gamma t} - \frac{1}{\sqrt{\pi\gamma t}} \left(\frac{2\gamma t}{1} + \frac{(2\gamma t)^2}{1 \cdot 3} + \frac{(2\gamma t)^3}{1 \cdot 3 \cdot 5} + \cdots \right) \right] \quad (84)$$

if we use the abbreviating notation for the series of the exponential. We could also demonstrate that this is, indeed, the expansion of (76) for small values of t by applying integration by parts to the error integral in the form (43) and by selecting now $u = e^{-y^2}$, $dv = dy$. However, we shall not carry through this demonstration.

The numerical computation with the convergent series expansion (84) does not permit a close estimate of the error committed if we break off at any one term. To illustrate this point, Fig. 7.8 shows the successive improvement in the value of the convergent series as more terms are included; the figure graphically demonstrates the *convergence* upon the exact value as contrasted with the oscillatory behavior of the asymptotic series.

In the foregoing example, the pole contributed by the second factor in the denominator of (73a) was of no significance for the solution of the problem. Let us take *another example* in which we need to take care of certain poles, namely the application of a *step voltage* to the noninductive cable *over the lumped inductance* L . The equivalent circuit is given in Fig. 7.9 from which we obtain for the Laplace transform of the input voltage similar to (72),

$$V(0,p) = \frac{Z_c}{Z_c + pL} \frac{V_m}{p} = \frac{V_m}{p} \sqrt{\frac{r}{c}} \frac{1}{\sqrt{p} \left(\sqrt{\frac{r}{cp}} + pL \right)}$$

If we introduce here $\gamma^{3/2} = (1/L) \sqrt{r/c}$ we can also write this more systematically

$$V(0, p) = V_m \frac{\gamma^{3/2}}{p(\gamma^{3/2} + p^{3/2})} \quad (85)$$

Since a “closed form” solution is not feasible, we might try to obtain the asymptotic behavior at both ends of the time scale, $t \rightarrow 0$ as well as

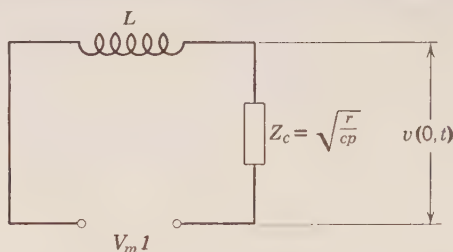


Fig. 7.9. Voltage applied to noninductive cable over a lumped inductance L .

$t \rightarrow \infty$. For the latter, we can separate (85) into two terms by inspection, namely

$$V(0, p) = V_m \left(\frac{1}{p} - \frac{p^{1/2}}{p^{3/2} + \gamma^{3/2}} \right) \quad (86)$$

It is obvious that the irrational part has a branch point at $p = 0$, contributed by the numerator, and therefore requires the same definition of corresponding regions in the p -plane and \sqrt{p} -plane as was discussed in connection with Fig. 7.2. But in addition we must realize that the denominator contributes three poles, defined by

$$p^{3/2} = -\gamma^{3/2}, \quad p^{1/2} = \gamma^{1/2}(-1)^{1/3}$$

involving the three unit roots of (-1) . Of course, we could have also stated the roots in terms of p as

$$p_{1,2,3} = \gamma e^{j2\pi/3}, \quad \gamma e^{j4\pi/3}, \quad \gamma e^{j6\pi/3}$$

However, in this selection, the first two roots which are conjugate complex lie in the two different branches, so that we would violate the principle that in every real physical network function complex poles must occur in conjugate pairs! The conventional choice above has not taken into account the fact that p as a complex number starts the values of its argument at $(-\pi)$ and goes to (3π) . The only choice of the three unit roots which is consistent with the choice of the branch cut along π and also retains the root values as given is then

$$p_{1,2,3} = \gamma e^{-j2\pi/3}, \quad \gamma e^{+j2\pi/3}, \quad \gamma e^{+j6\pi/3} = \gamma$$

so that in turn

$$(\sqrt{p})_{1,2,3} = \sqrt{\gamma} e^{-j\pi/3}, \quad \sqrt{\gamma} e^{+j\pi/3}, \quad \sqrt{\gamma} e^{j\pi} \quad (87)$$

It is obvious that the last value is $-\sqrt{\gamma}$, which cubed, gives $-\gamma^{3/2}$ as required, and so do the first two values. Since we had originally chosen the branch $\sqrt{p} > 0$ for the physical solutions (10), (13), etc., we see that the first two, conjugate complex, poles

$$p_{1,2} = \left(-\frac{1}{2} \mp j \frac{\sqrt{3}}{2} \right) \gamma \quad (88)$$

must be taken into account here, whereas the third pole can be disregarded as lying in the branch that is of no interest.

We are now in a position to approximate (86) asymptotically as $t \rightarrow \infty$. The rational first term gives in accordance with (70) the value

$$\lim_{p \rightarrow 0} p \frac{1}{p} \rightarrow 1$$

The residues for the conjugate complex pair of poles of the irrational part are found as for any analytic function, since we have removed the double valuedness. Defining the second term in (86) conventionally

$$\frac{p^{1/2}}{p^{3/2} + \gamma^{3/2}} = \frac{N(p)}{D(p)}$$

we have for the sum of the two residues

$$\begin{aligned} \sum_{p=p_{1,2}} \left[\frac{N(p)}{[d/(dp)]D(p)} e^{pt} \right] &= \frac{2}{3} e^{-\frac{1}{2}\gamma t} (e^{j\frac{\sqrt{3}}{2}\gamma t} + e^{-j\frac{\sqrt{3}}{2}\gamma t}) \\ &= \frac{4}{3} e^{-\frac{1}{2}\gamma t} \cos \left(\frac{\sqrt{3}}{2} \gamma t \right) \end{aligned}$$

Finally, we expand this same second term into an ascending power series in \sqrt{p}

$$\begin{aligned} \frac{p^{1/2}}{p^{3/2} + \gamma^{3/2}} &= \frac{p^{1/2}}{\gamma^{3/2}} \frac{1}{1 + \left(\frac{p}{\gamma} \right)^{3/2}} \\ &= \sqrt{\frac{p}{\gamma^3}} \left[1 - \left(\frac{p}{\gamma} \right)^{3/2} + \left(\frac{p}{\gamma} \right)^{6/2} - \left(\frac{p}{\gamma} \right)^{9/2} + \cdots \right] \end{aligned}$$

in which we drop the resultant integer powers of p and find the inverse Laplace transform of the remaining terms by pairs 2 and 3, Table 7.1

$$- \frac{1}{\sqrt{\pi\gamma t}} \left(\frac{1}{2\gamma t} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{(2\gamma t)^4} + \frac{1 \cdot 3 \cdot 5 \cdot \cdot \cdot 13}{(2\gamma t)^7} - \cdot \cdot \cdot \right)$$

The total asymptotic approximation for the inverse transform of (86) for large values of t is therefore

$$v(0,t) = V_m \left[1 - \frac{4}{3} e^{-\frac{1}{2}\gamma t} \cos \left(\frac{\sqrt{3}}{2} t \right) + \frac{1}{\sqrt{\pi\gamma t}} \left(\frac{1}{2\gamma t} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{(2\gamma t)^4} + \cdot \cdot \cdot \right) \right] \quad (89)$$

It is important to point to the trigonometric function in (83) indicating an overshoot in the response because of the combination of inductance and capacitance in series, even though the capacitance has an irrational frequency dependence. In earlier treatments of these problems, which refer back to Heaviside, makeshift arguments had to be introduced to obtain the complete solution, e.g., Bush, *op. cit.*, chapter XII, pp. 242-255. The convergent approximation for $t \rightarrow 0$ can be obtained much more readily and is also found in the reference.

7.6 Response of the Infinite Cable to a Sinusoidal Voltage

The Laplace transform for the voltage distribution along the line is given by (10) for any general source voltage $v_0(t)$. If we apply a sinusoidal source voltage

$$v_0(t) = V_m \sin \omega t$$

with the transform pair 12 in Table 1.3, we have

$$V(x,p) = V_m \frac{\omega}{p^2 + \omega^2} e^{-\sqrt{ap}} \quad \text{Re } \sqrt{p} > 0 \quad (90)$$

where we used $a = rcx^2$. The inverse Laplace transform cannot be found directly in "closed form" so we shall have to invoke series expansions as, e.g., in McLachlan,^{D13} pp. 91-95.

Perhaps the nearest approach to a closed form solution is by means of the convolution integral pair 11 of Table 1.4. We define

$$\begin{aligned} F_1(p) &= \frac{\omega}{p^2 + \omega^2} & f_1(t) &= \sin \omega t \\ F_2(p) &= e^{-\sqrt{ap}} & f_2(t) &= \frac{1}{\sqrt{\pi t}} \sqrt{\frac{a}{4t}} \exp \left(-\frac{a}{4t} \right) \end{aligned}$$

where the second inverse transform is taken from (38), or pair 11, Table 7.1. Application of the convolution integral gives

$$v(x, t) = V_m \sqrt{\frac{a}{4\pi}} \int_{\tau=0}^t \sin \omega(t - \tau) \exp\left(-\frac{a}{4\tau}\right) \frac{d\tau}{\tau \sqrt{\tau}} \quad (91)$$

If we expand the trigonometric function and use as the variable of integration $u = (4\tau)/a$, we can write (91) in the simple form

$$\frac{v(x, t)}{V_m} = M_c\left(\Omega, \frac{4t}{a}\right) \sin \omega t - M_s\left(\Omega, \frac{4t}{a}\right) \cos \omega t \quad (92)$$

with the coefficients defined by the integrals

$$M_c\left(\Omega, \frac{4t}{a}\right) = \frac{1}{\sqrt{\pi}} \int_{u=0}^{\frac{4t}{a}} \frac{\cos \Omega u}{u \sqrt{u}} e^{-\frac{1}{u}} du \quad (93)$$

$$M_s\left(\Omega, \frac{4t}{a}\right) = \frac{1}{\sqrt{\pi}} \int_{u=0}^{\frac{4t}{a}} \frac{\sin \Omega u}{u \sqrt{u}} e^{-\frac{1}{u}} du$$

with $\Omega = (a\omega)/4$, so that $\Omega u = \omega t$. The functions M_c and M_s have the meaning of transient amplitude modulations of the same general form as the Fresnel integrals. In fact, for $0 < u < 2$ we can roughly replace

$$e^{-\frac{1}{u}} \approx 0.3u$$

so that with $y = \omega t$

$$M_c = 0.3 \sqrt{\frac{2}{\Omega}} \frac{1}{\sqrt{2\pi}} \int_{y=0}^{y=\omega t} \frac{\cos y}{\sqrt{y}} dy = \frac{0.6 \sqrt{2}}{\sqrt{a\omega}} C(\omega t) \quad (94)$$

and similarly

$$M_s = \frac{0.6 \sqrt{2}}{\sqrt{a\omega}} S(\omega t) \quad (94a)$$

where now C and S are the standard Fresnel integrals which are tabulated.* We have approximately

$$v(x, t) = \frac{0.6 \sqrt{2}}{x \sqrt{rc\omega}} V_m [C(\omega t) \sin \omega t - S(\omega t) \cos \omega t]_{t < a/2} \quad (95)$$

which, for all practical purposes, covers the transient build-up period. Fig. 7.10 shows graphs of the applied voltage $v(0, t)$ which is, of course,

* Jahnke-Emde, *op. cit.*, pp. 35-36.

also the voltage $v(x, t)$ at $x = 0$, and of the voltage (95) at a distance $x \geq (0.6 \sqrt{2})/\sqrt{r\omega c}$. We observe a definite unsymmetrical voltage variation with time which is also borne out by the asymptotic expansion.

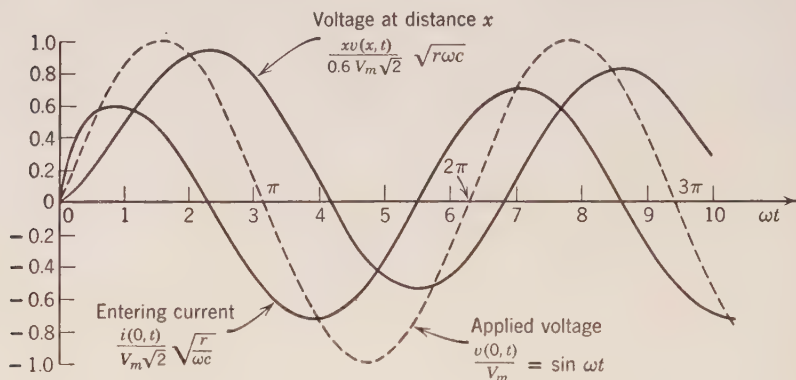


Fig. 7.10. Response of infinite cable to sinusoidal voltage.

For large values of time we have to expand in accordance with (71), first separating the residues at complex poles. As (90) indicates, there are the two poles $p = \pm j\omega$ which have the images in the \sqrt{p} plane

$$p = +j\omega, \quad \sqrt{p} = +\sqrt{j}\sqrt{\omega} \rightarrow +\frac{1+j}{\sqrt{2}}\sqrt{\omega}$$

$$p = -j\omega, \quad \sqrt{p} = -j\sqrt{j}\sqrt{\omega} \rightarrow +\frac{1-j}{\sqrt{2}}\sqrt{\omega}$$

The selection of the values for \sqrt{p} is facilitated if we again observe that the argument of p in the first branch runs from $-\pi$ to $+\pi$ so that we must interpret $\pm j = e^{\pm j\pi/2}$ which leads at once to the proper conjugate complex choices of \sqrt{p} . The residues at these two poles are from (90), respectively

$$V_m \frac{\omega}{p \pm j\omega} e^{-\sqrt{ap}} e^{pt} \Big|_{p=\pm j\omega} = V_m \frac{\pm 1}{2j} \exp\left(-\sqrt{a}\frac{\omega}{2} \mp j\sqrt{a}\frac{\omega}{2}\right) e^{\mp j\omega t}$$

and the sum of these two terms leads to

$$V_m e^{-\sqrt{a}\frac{\omega}{2}} \sin\left(\omega t - \sqrt{\frac{a\omega}{2}}\right) \quad (96)$$

We next need to expand (90) into ascending powers of \sqrt{p} for small values of p . The exponential gives an absolutely convergent series,

and so does the rational fraction of p by the binominal series for small p , so that we can use the product of the expansions. Restricting ourselves to the first few terms, we have simply

$$\frac{\omega}{p^2 + \omega^2} e^{-\sqrt{ap}} = \frac{1}{\omega} \left[1 - \left(\frac{p}{\omega} \right)^2 + \left(\frac{p}{\omega} \right)^4 - \dots \right] \left(1 - \sqrt{ap} + \frac{ap}{2} - \dots \right)$$

As shown in (71), we drop the integer powers and have as first term of the half power series from pair 2 of Table 7.1

$$\mathcal{L}^{-1} \left(-\frac{1}{\omega} \sqrt{ap} \right) = + \frac{\sqrt{a}}{\omega} \frac{1}{2 \sqrt{\pi} t^{3/2}} = \frac{\sqrt{r\omega c} x}{2 \sqrt{\pi} (\omega t)^{3/2}}$$

This with (96) gives the total asymptotic expansion for $t \rightarrow \infty$

$$v(x, t) = V_m \left[e^{-x \sqrt{\frac{r\omega c}{2}}} \sin \left(\omega t - x \sqrt{\frac{r\omega c}{2}} \right) + \frac{\sqrt{r\omega c} x}{2 \sqrt{\pi} (\omega t)^{3/2}} \right] \quad (97)$$

It is this last term which causes the unsymmetry in the voltage wave as is evident in Fig. 7.10.

For small values of time and distance, (95) is not accurate because of the rather coarse approximation to the exponential $e^{-1/u}$. We can improve this approximation very considerably if we replace the exponential sectionally as follows

$$\begin{aligned} e^{-\frac{1}{u}} &\approx 0.5u^2 & 0 \leq u \leq 0.5 \\ &0.25u & 0.5 \leq u \leq 2 \\ &1 - \frac{1}{u} & 2 \leq u < \infty \end{aligned} \quad (98)$$

and perform the integrations (93) also sectionally. All the integrals can be reduced by integration by parts to trigonometric functions and the standard Fresnel integrals so that no new problems will arise. For very small values of time, we can also expand the rational factor in (90) into the convergent binominal series valid for p very large

$$\frac{\omega}{p^2 + \omega^2} e^{-\sqrt{ap}} = \frac{\omega}{p^2} \left[1 - \left(\frac{\omega}{p} \right)^2 + \left(\frac{\omega}{p} \right)^4 - \dots \right] e^{-\sqrt{ap}}$$

Taking the first term only, we find

$$\mathcal{L}^{-1} \frac{\omega}{p^2} e^{-\sqrt{ap}} = \omega \mathcal{L}^{-1} \frac{1}{p} \left(\frac{1}{p} e^{-\sqrt{ap}} \right)$$

The term within the parenthesis is readily identified from pair 15, Table 7.1, and the factor $1/p$ indicates integration by pair 5a, Table 1.4, so that asymptotically for $t \rightarrow 0$

$$v(x, t) = \omega V_m \int_0^t \operatorname{erfc} \left(\sqrt{\frac{a}{4t}} \right) dt \quad (99)$$

which can be evaluated graphically from Fig. 7.4. In any case, it demonstrates the initial rise of the voltage shown in Fig. 7.10.

We might finally solve for the *entering current* of the cable in order to compare this a-c response with the response to a step voltage given in (21a). The Laplace transform of the current is given by (11) for any general source function $v_0(t)$. If we choose again the sinusoidal source voltage $V_m \sin \omega t$, then we obtain at $x = 0$

$$I(0, p) = \frac{V_m}{\sqrt{r/c}} \frac{\omega \sqrt{p}}{p^2 + \omega^2} = \frac{V_m}{\sqrt{r/c}} \frac{\omega p}{\omega^2 + p^2} \left(\frac{1}{p} \sqrt{p} \right) \quad (100)$$

The last form of (100) makes the expression ready for the application of the convolution integral, pair 11 in Table 1.4, if we define

$$F_1(p) = \frac{p}{\omega^2 + p^2} \quad f_1(t) = \cos \omega t$$

$$F_2(p) = \frac{1}{\sqrt{p}} \quad f_2(t) = \frac{1}{\sqrt{\pi t}}$$

The complete solution for the inverse Laplace transform of (100) is then

$$i(0, t) = \frac{V_m}{\sqrt{r/c}} \omega \int_{\tau=0}^t \frac{1}{\sqrt{\pi \tau}} \cos \omega(t - \tau) d\tau \quad (101)$$

If we expand the trigonometric function and introduce $\omega\tau = y$ as a new variable, we can reduce (101) again to the standard Fresnel integrals, namely

$$\frac{1}{\sqrt{\pi}} \int_{\tau=0}^t \frac{1}{\sqrt{\tau}} \cos \omega\tau d\tau = \sqrt{\frac{2}{\omega}} \frac{1}{\sqrt{2\pi}} \int_{y=0}^{y=\omega t} \frac{1}{\sqrt{y}} \cos y dy = \sqrt{\frac{2}{\omega}} C(\omega t)$$

and similarly for the $\sin \omega t$ term.

We thus have

$$i(0,t) = \frac{V_m \sqrt{2}}{\sqrt{r/\omega c}} [C(\omega t) \cos \omega t + S(\omega t) \sin \omega t] \quad (102)$$

which is shown graphically in Fig. 7.10. We see that the current reaches its maximum ahead of the applied voltage as in any capacitive circuit. The final steady-state solution can readily be found by taking the residues of (100) augmented by the factor e^{pt} at the two poles $p = \pm j\omega$, as we had done for the voltage in finding its asymptotic expression (96). We have here

$$\left. \frac{V_m}{\sqrt{r/c}} \frac{\omega \sqrt{p}}{p \pm j\omega} e^{pt} \right|_{p=\pm j\omega} = \frac{V_m}{\sqrt{r/c}} \frac{\pm 1}{2j} \frac{1 \pm j}{\sqrt{2}} \omega e^{\pm j\omega t}$$

and the sum of these residues gives the steady-state expression

$$i(0,t = \infty) = \frac{V_m}{\sqrt{r/c}} \sqrt{\frac{\omega}{2}} (\cos \omega t + \sin \omega t) = \frac{V_m}{\sqrt{r/\omega c}} \sin (\omega t + \pi/4) \quad (103)$$

indicating an ultimate leading phase angle of $\pi/4 = 45^\circ$ with respect to the applied voltage. We can rationalize this in terms of the $\sqrt{\omega}$ -dependence of the characteristic impedance of the cable in (5) where we find for the conventional a-c impedance

$$Z(j\omega) = \sqrt{\frac{r}{j\omega c}} = \sqrt{\frac{r}{\omega c}} e^{-j\pi/4}$$

Computation in terms of the phasors would have led to the same result (103).

The initial rate of rise of the entering current is infinitely large. We can get the asymptotic expression for the current for $t \rightarrow 0$ by expanding the rational fraction in (100) into the binominal series valid for p very large, so that

$$\frac{\sqrt{p}}{p^2 + \omega^2} = \frac{\sqrt{p}}{p^2} \frac{1}{1 + \left(\frac{\omega}{p}\right)^2} = p^{-3/2} \left[1 - \left(\frac{\omega}{p}\right)^2 + \left(\frac{\omega}{p}\right)^4 - \dots \right]$$

Keeping the first term only, we have the initial variation of the current for t small

$$i(0,t) = \frac{V_m}{\sqrt{r/c}} \omega \mathcal{L}^{-1} \frac{1}{p^{3/2}} = \frac{2}{\sqrt{\pi}} V_m \frac{\sqrt{\omega t}}{\sqrt{r/\omega c}} \quad (104)$$

The rate of rise at $t = 0$

$$\left(\frac{di(0,t)}{dt}\right)_{t \rightarrow 0} \text{prop} \frac{1}{\sqrt{t}}$$

becomes actually infinite and has, indeed, the same general form as (21a), the current response to a step voltage.

We have considered only the voltage form $V_m \sin \omega t$. It is quite obvious that if we chose $V_m \cos \omega t$, the first instant of response must be identical with that in the case of an applied step voltage.

7.7 Infinitely Long Cable With Slight Leakage

The ideally insulated cable has been characterized by the two parameters r, c which we have admitted in (1) as a special case of the general transmission line equations (6.21). This has led to the introduction of the branch cut, Fig. 7.3, in the complex plane defining a new path of integration for the inverse Laplace transform.

If we now also admit leakance g in the cable, we must once more return to the general equations (6.21) which take the form

$$\begin{aligned} -\frac{\partial V(x,p)}{\partial x} &= rI(x,p) \\ -\frac{\partial I(x,p)}{\partial x} &= (g + pc)V(x,p) - cv(x,0) \end{aligned} \quad (105)$$

Again, $v(x,0)$ is the initial distribution of the voltage on the cable which must be given as part of a specific problem; we shall generally disregard it. The two first-order differential equations for the Laplace transforms combine by the same process as used for (2) into

$$\frac{\partial^2 V(x,p)}{\partial x^2} = r(g + cp)V(x,p) = n^2 V(x,p) \quad (106)$$

Thus we have the propagation function

$$n = \sqrt{r(g + pc)} = \sqrt{rc \left(p + \frac{g}{c}\right)} \quad (107)$$

This is again irrational and requires the careful definition of the analytic region for the integrands in the inverse Laplace transforms. To be specific, let us take the infinitely long leaky cable and apply a step voltage $V_m t$ with Laplace transform V_m/p . The general current and voltage solution for the cable can be taken from (3) and (4)

with the value of n as specified in (107). If we also define the characteristic impedance

$$Z_c = \frac{r}{n} = \sqrt{\frac{r}{c(p + g/c)}} \quad (108)$$

we can write the specific Laplace transform for the infinitely long cable directly from (10) and (11), namely

$$V(x, p) = \frac{V_m}{p} e^{-nx} = \frac{V_m}{p} e^{-\sqrt{rc(p+g/c)}x} \quad (109)$$

and

$$I(x, p) = \frac{V_m}{pZ_c} e^{-nx} = \frac{V_m}{p\sqrt{r}} \sqrt{c(p + g/c)} e^{-\sqrt{rc(p+g/c)}x} \quad (110)$$

With $g = 0$ we had defined the useful range of \sqrt{p} as $\text{Re } \sqrt{p} > 0$ in order to obtain attenuation along the line rather than the physically impossible generation of voltage and current. This had led to the selection of branch 1 in (8) with the concomitant definition (7) for the range of the arguments of p . The additive quantity g/c in (107) is certainly always positive real, so that $\text{Re } \sqrt{p + g/c}$ will be even more positive than $\text{Re } \sqrt{p}$ and we can maintain the same selection of branch 1 of \sqrt{p} for the Laplace transform functions (109) and (110).

Let us first solve for the entering current of the leaky cable. From (110) we have with $g/c = \delta$

$$I(0, p) = \frac{V_m}{\sqrt{r/c}} \frac{1}{p} \sqrt{p + g/c} = \frac{V_m}{\sqrt{r/c}} \frac{p + \delta}{p \sqrt{p + \delta}} \quad (111)$$

which permits the separation

$$\frac{1}{\sqrt{p + \delta}} + \frac{\delta}{p \sqrt{p + \delta}}$$

The inverse Laplace transform of the square root is rather simple if we observe pair 6 in Table 1.4 which gives

$$\mathcal{L}^{-1} \frac{1}{\sqrt{p + \delta}} = e^{-\delta t} \mathcal{L}^{-1} \frac{1}{\sqrt{p}} = + \frac{e^{-\delta t}}{\sqrt{\pi t}} \quad (112)$$

where we called on Table 7.1, pair 2, for the last form. By pair 5a, Table 1.4, we can interpret the factor $1/p$ so that we get

$$\frac{\delta}{p \sqrt{p + \delta}} = \delta \int_{0+}^t \frac{e^{-\delta t}}{\sqrt{\pi t}} dt = \sqrt{\delta} \frac{2}{\sqrt{\pi}} \int_0^y e^{-y^2} dy \quad (113)$$

Here, we introduced $\delta t = y^2$, $dt = (2/\delta)y dy$, so as to obtain again the error integral similar to the one used in (75). We can collect now from (112) and (113)

$$i(0,t) = \frac{V_m}{\sqrt{r/c}} \left(\frac{e^{-\delta t}}{\sqrt{\pi t}} + \sqrt{\delta} \operatorname{erf} \sqrt{\delta t} \right) \quad (114)$$

This goes over into (21a) as it should if we let $\delta \rightarrow 0$. The effect of the leakance is then a quicker decay of the first or "charging" part of the current because of the added exponential factor, and the establishment of a steady-state d-c leakage current defined by the second term as

$$i(0,t \rightarrow \infty) \rightarrow V_m \sqrt{\frac{g}{r}}$$

and directly dependent upon the leakance g . We should, of course, be able to obtain this result also directly from the asymptotic form given as pair 23 of Table 1.4

$$\lim_{t \rightarrow \infty} i(0,t) = \lim_{p \rightarrow 0} [pI(0,p)] = \frac{V_m}{\sqrt{r/c}} \sqrt{g/c}$$

In a similar manner, we can find the inverse time functions to the Laplace transforms (109) and (110). For the voltage transform we can write with $a = rcx^2$ as previously

$$\begin{aligned} V(x,p) &= \frac{V_m}{p} (p + \delta) \left[\frac{1}{p + \delta} e^{-\sqrt{a(p+\delta)}} \right] \\ &= \left(1 + \frac{\delta}{p} \right) \left(\frac{V_m}{p + \delta} e^{-\sqrt{a(p+\delta)}} \right) \quad (115) \end{aligned}$$

where the expression in brackets is the Laplace transform (10) with p replaced by $(p + \delta)$. If we designate here the voltage distribution for $g = 0$ (no leakance) by the subscript 0, then we can write by proper application of the shifting theorem, pair 6, Table 1.4

$$\mathcal{L}^{-1} \frac{V_m}{p + \delta} e^{-\sqrt{a(p+\delta)}} = e^{-\delta t} v_0(x,t)$$

Thus we obtain the inverse transform of (115) in the simple and physically easily interpretable form

$$v(x,t) = e^{-\delta t} v_0(x,t) + \delta \int_{t=0}^t e^{-\delta t} v_0(x,t) dt \quad (116)$$

The function $v_0(x,t)$ is identical with (45) and shown in Fig. 7.4, namely the response of the nonleaky cable to a step source voltage.

The effect of leakance is therefore the appearance of the damping factor $e^{-\delta t}$ with $\delta = g/c$ and the addition of the integral term which is usually a small correction because g is a small quantity. Of course, this integration is difficult and normally has to be performed graphically or numerically.

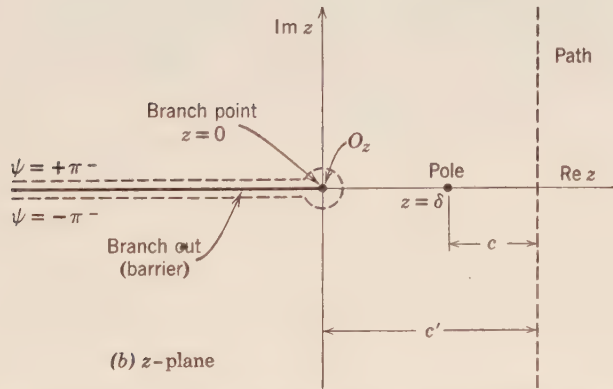
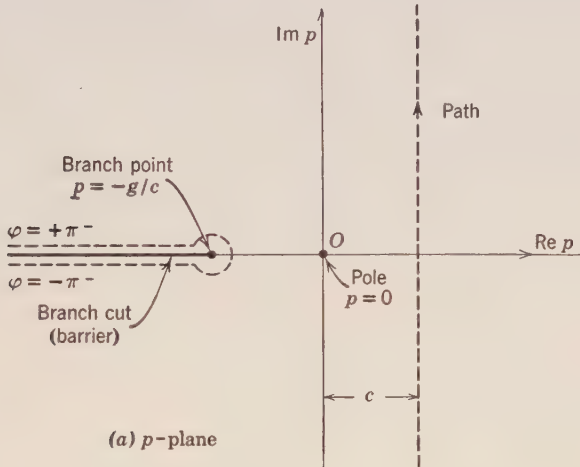


Fig. 7.11. Singularities of the voltage Laplace transform for the leaky cable. (a) In the p -plane; (b) in the z -plane (branch point at origin).

So far we have only used relationships which needed no detail examination of the function theoretical aspects of the Laplace transforms themselves. Let us return to (109) and examine its singularities. The branch point, characteristic of the exponential, is

defined as $p = g/c$ and is located on the negative real axis as shown in Fig. 7.11a. The branch cut will therefore terminate at this point. In addition, (109) has a pole at $p = 0$, which is now distinct from the branch point and falls within the first branch of the p -plane which we are using in our solution. We can make this clearer if we perform a shift of the origin by introducing a new complex variable $z = p + g/c = p + \delta$, so that (109) becomes

$$V(x, z) = \frac{V_m}{z - \delta} e^{-\sqrt{az}} \quad (117)$$

In this z -plane shown in Fig. 7.11b, the origin $z = 0$ is again the branch point and we can use exactly the methods demonstrated in sections 7.1 and 7.2 to perform the integrations for the evaluation of the inverse Laplace transform, which is now defined by the integral

$$v(x, t) = \frac{1}{2\pi j} e^{-\delta t} \int_{c' - j\infty}^{c' + j\infty} \frac{V_m}{z - \delta} e^{-\sqrt{az} e^{zt}} dz \quad (118)$$

The factor $e^{-\delta t}$ outside the integral comes from transforming e^{pt} into $e^{(z-\delta)t}$, and since δ is a constant the exponential can be split to leave the standard factor e^{zt} inside the integral. We also must shift the path of integration further to the right since the pole $z = \delta$ is now on the *positive* real z -axis. We could readily demonstrate the inverse Laplace transform in the same form as (116) by just expanding the integrand (118) to

$$V_m \frac{z}{z - \delta} \frac{1}{z} e^{-\sqrt{az}} = \frac{z}{z - \delta} V_0(x, p)$$

where $V_0(x, p)$ is the transform (10) for the nonleaky cable and then applying the convolution integral.

Instead, we shall use (118) to derive the asymptotic expression for large values of t in accordance with section 7.4. It is not possible to separate a rational part of the integrand, but we have to take note of the pole $z = \delta$. The residue is readily given by

$$\text{Res} \left[\frac{1}{z - \delta} e^{-\sqrt{az} e^{zt}} \right]_{z=\delta} = e^{-\sqrt{a\delta} e^{\delta t}} \Big|_{z=\delta} = e^{-\sqrt{a\delta} e^{\delta t}}$$

The expansion of the exponential function $e^{-\sqrt{az}}$ into ascending powers of \sqrt{z} is an absolutely convergent series and the binominal series for $(z - \delta)^{-1}$ is also convergent for $|z/\delta| < 1$ so that for small values of z we can multiply the two expansions. Ordering as in (71), we arrive at

$$\frac{1}{z-\delta}e^{-\sqrt{az}}=-\frac{1}{\delta}\left[1-\frac{z}{\delta}+\frac{az}{2!}+\left(\frac{z}{\delta}\right)^2-\frac{az^2}{\delta 2!}+\cdots\right]$$
$$+\frac{1}{\delta}\sqrt{az}\left(1-\frac{z}{\delta}+\frac{az}{3!}-\cdots\right)$$

We drop the integer power series at once and for very large values of t restrict our attention to the very first term of the half power series. This gives with pair 2 of Table 7.1

$$\mathcal{L}^{-1}\frac{1}{\delta}\sqrt{az}=-\frac{\sqrt{a}}{\delta}\frac{1}{2\sqrt{\pi}}\frac{1}{t^{\frac{3}{2}}}=-\frac{1}{2\sqrt{\pi}}\frac{\sqrt{az}}{(\delta t)^{\frac{3}{2}}}$$

The complete asymptotic approximation for (118) at $t\rightarrow\infty$ is now

$$v(x,t)=e^{-\delta t}V_m\left(e^{-\sqrt{a\delta}e^{\delta t}}-\frac{1}{2\sqrt{\pi}}\frac{\sqrt{a\delta}}{(\delta t)^{\frac{3}{2}}}\right)$$

and if we introduce

$$a\delta=rcx^2\frac{g}{c}=rgx^2$$

we obtain

$$v(x,t)=V_m\left(e^{-\sqrt{rg}x}-\frac{1}{2}\sqrt{rg}x\frac{e^{-\delta t}}{(\delta t)^{\frac{3}{2}}}\right)\tag{119}$$

Because of the leakance g , there will be a final exponential voltage distribution along the cable given by the first term in (119). We can apply the same principle to get the inverse transform of (110), the Laplace transform of the current.

7.8 The Finite Noninductive Cable

The finite cable terminated at both ends into lumped-element impedances as indicated in Fig. 7.12 requires return to the general

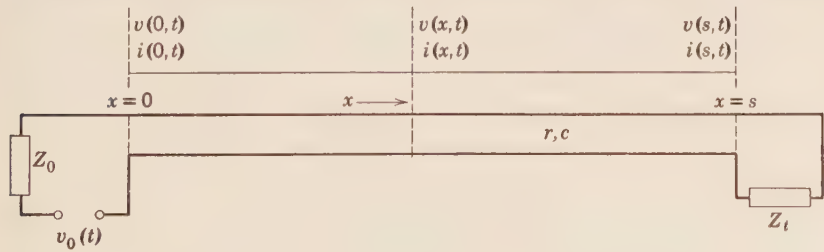


Fig. 7.12. The terminated noninductive cable.

solutions (3) and (4) for the Laplace transforms of voltage and current. Indeed, since we are using there n , the propagation function, and Z_c ,

the parametric characteristic impedance, without immediate specification of their dependences on p , we can go back further to section 6.3 and find in (6.46) the complete statement of the boundary conditions at both ends of the cable, in (6.47) the convenient definitions of the current reflection coefficients

$$\rho_0 = \frac{Z_c - Z_0}{Z_c + Z_0}, \quad \rho_t = \frac{Z_c - Z_t}{Z_c + Z_t} \quad (120)$$

and in (6.48), (6.49) the complete solutions for the Laplace transforms of current and voltage. We might repeat these here for the simpler case that $Z_0 = 0$, so that we apply the source voltage or current directly to the input terminals of the cable. With $\rho_0 = 1$, we therefore have

$$V(x, p) = \frac{e^{-nx} - \rho_t e^{-n(2s-x)}}{1 - \rho_t e^{-2ns}} \mathfrak{L}v_0(t) \quad (121)$$

$$I(x, p) = \frac{e^{-nx} + \rho_t e^{-n(2s-x)}}{1 - \rho_t e^{-2ns}} \frac{\mathfrak{L}v_0(t)}{Z_c} \quad (122)$$

We could now proceed as in section 6.3 and expand the fractions into an exponential series, e.g., for the voltage

$$V(x, p) = \mathfrak{L}v_0(t) [e^{-nx} - \rho_t e^{-n(2s-x)} + \rho_t e^{-n(2s+x)} - \rho_t^2 e^{-n(4s-x)} + \cdots] \quad (123)$$

where again the first term is identical with the Laplace transform for the infinitely long cable given in (10). We actually have solved (10) for a step source voltage and found the inverse Laplace transform in (45). This means that we can also write down at once the inverse Laplace transforms of all the exponentials that occur in (123), if $v_0(t) = V_m I$, namely

$$\begin{aligned} \mathfrak{L}^{-1} \frac{1}{p} e^{-n(2\nu s \pm x)} &= \mathfrak{L}^{-1} \frac{1}{p} e^{-\sqrt{rc}(2\nu s \pm x)\sqrt{p}} \\ &= \operatorname{erfc} \left(\frac{1}{2} (2\nu s \pm x) \sqrt{\frac{rc}{t}} \right) \end{aligned} \quad (124)$$

where ν was chosen as any integer from zero up. However, we must observe that each and every one of these terms starts at $t = 0$, and that there is no special physical meaning connected with this expansion such as we had in section 6.3 where the individual terms represented real physical traveling waves, successively generated at the ends of the line by the process of reflection! Series (123), although appearing

similar to the traveling wave expansion (6.50), has no such meaning attached to it! It is true that the individual inverse time functions might exhibit an apparent delay because the $\text{Re} [(2\nu s \pm x) \sqrt{p}]$ grow larger as ν increases; yet at any instant we need to consider the infinite series of terms in order to represent $v(x, t)$. We must also note that, although (124) is easily written, the inverse Laplace transforms of all but the first term involve first or higher powers of the reflection factor ρ_t . As (120) shows, if we introduce Z_c from (5), we have again irrational forms

$$\rho_t = \frac{\sqrt{r/c} - Z_t(p) \sqrt{p}}{\sqrt{r/c} + Z_t(p) \sqrt{p}} \quad (125)$$

where $Z_t(p)$, as we assumed, is some rational function of p . Even for simple terminal impedances the task of converting (123) into the inverse time functions appears rather formidable and of no particular advantage over (121) itself.

We might therefore turn to the other method which we discussed in section 6.9, namely the expansion into standing waves by direct application of the theorem of residues. Of course, this requires a careful examination of (121) as a function of the complex variable p . We might start with the same special case as in section 6.9, namely the short-circuited line, so that we can compare the results for these two different types of transmission lines. We have $Z_t = 0$, $\rho_t = 1$ so that the voltage transform follows from (121), quite similar to (6.83) for the lossless transmission line

$$V(x, p) = \frac{\sinh n(s - x)}{\sinh ns} \mathcal{L}v_0(t) = \frac{\sinh [\sqrt{rc} (s - x) \sqrt{p}]}{\sinh [\sqrt{rc} s \sqrt{p}]} \mathcal{L}v_0(t) \quad (126)$$

First of all we must demonstrate the admissibility of Cauchy's residue theorem. Though (126) has in numerator and denominator an irrational transcendental function, each can be written in terms of its convergent power series, e.g.

$$\sinh z = z \left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots \right)$$

or also, as already used in section 3.2, in terms of the root factors as a meromorphic function

$$\sinh z = z \left[1 + \left(\frac{z}{\pi} \right)^2 \right] \left[1 + \left(\frac{z}{2\pi} \right)^2 \right] \left[1 + \left(\frac{z}{3\pi} \right)^2 \right] \cdots \quad (127)$$

We see that in either case the \sqrt{p} will cancel and the fraction will become entirely rational so that we can apply at once the expansion into the partial fractions indicated by (127). The fact that we deal here with an infinite series of root factors and therefore an infinite sum of residues has been discussed both in sections 3.2 and 6.9 and we simply refer there for the justifying arguments.

If we replace the variable $z = (\sqrt{rc} s) \sqrt{p} = \sqrt{bp}$ in (127), disregard the first factor because of the cancellation of \sqrt{p} , we find the root values

$$p_\alpha = -\frac{(\alpha\pi)^2}{rcs^2} = -\frac{(\alpha\pi)^2}{b} \quad \alpha = 1, 2, \dots \quad (128)$$

all negative real as we should expect from the physical aspects of the noninductive cable. We may carry through the partial fraction expansion without as yet specifying the source voltage, by using the general expression (1.55)

$$\frac{N(p)}{D(p)} = \frac{\sinh(\sqrt{b} - \sqrt{a}) \sqrt{p}}{\sinh \sqrt{bp}} = \sum_{\alpha=1}^{\infty} \left(\frac{N(p)}{dD(p)/dp} \right)_{p=p_\alpha} \frac{1}{p - p_\alpha}$$

The derivative of the denominator gives

$$\frac{d}{dp} \sinh \sqrt{bp} = \frac{1}{2} \sqrt{\frac{b}{p}} \cosh \sqrt{bp}$$

If we expand the numerator

$$\sinh(\sqrt{b} - \sqrt{a}) \sqrt{p} = \sinh \sqrt{bp} \cosh \sqrt{ap} - \cosh \sqrt{bp} \sinh \sqrt{ap}$$

and observe that the first term vanishes at all root values because $(\sinh \sqrt{bp})_{p_\alpha} = 0$, the summation reduces to

$$\sum_{\alpha=1}^{\infty} (-2) \left(\sqrt{\frac{p}{b}} \sinh \sqrt{ap} \right)_{p=p_\alpha} \frac{1}{p - p_\alpha} \quad (129)$$

Here we must introduce p_α from (128) selecting $(-1) = e^{j\pi}$ so that

$$\sqrt{p_\alpha} = j \frac{\alpha\pi}{\sqrt{b}} = j \frac{\alpha\pi}{\sqrt{rc} s}, \quad \sinh \sqrt{ap_\alpha} = j \sin \left(\alpha\pi \frac{x}{s} \right)$$

The total expansion of the voltage (126) is thus

$$V(x, p) = \frac{2\pi}{rcs^2} \sum_{\alpha=1}^{\infty} \alpha \sin \left(\alpha\pi \frac{x}{s} \right) \frac{\mathcal{L}v_0(t)}{p - p_\alpha} \quad (130)$$

We recognize this as the resolution of the voltage distribution into a spatial Fourier series exactly like (6.87) for the lossless transmission line. The amplitudes are again functions of time, dependent only upon the nature of the source function! The fundamental wave length for $\alpha = 1$ is $\lambda_1 = 2s$, twice the length of the cable. This justifies calling (130) an expansion into "standing waves" with fixed individual sinusoidal distributions along the line. The time variation of the amplitudes will result in a variable over-all pattern of the voltage distribution, leading from the original zero voltage to the final steady state.

For the step source voltage $v_0(t) = V_m t$ with Laplace transform V_m/p we could either proceed as in section 6.9 by finding the inverse of (130) or, and better, by returning to (126) and applying to it directly the conventional expansion theorem (1.57) giving the indicial transfer function. For $p = 0$, the hyperbolic fraction reduces to

$$\lim_{p \rightarrow 0} \frac{\sinh(\sqrt{b} - \sqrt{a}) \sqrt{p}}{\sinh \sqrt{bp}} = \frac{\sqrt{b} - \sqrt{a}}{\sqrt{b}} = 1 - \frac{x}{s}$$

because of (127). The amplitudes $\left[\frac{N(p)}{p[dD(p)/dp]} \right]_{p\alpha}$ can be taken from (129) if we multiply there by $p\alpha^{-1}$. The total result will therefore be

$$v(x, t) = V_m \left[\left(1 - \frac{x}{s} \right) - \frac{2}{\pi} \sum_{\alpha=1}^{\infty} \frac{1}{\alpha} \sin \left(\alpha \pi \frac{x}{s} \right) e^{-\frac{(\alpha \pi)^2 t}{rcs^2}} \right] \quad (131)$$

The first term is the linear voltage drop along the cable under steady-state conditions, and the second term constitutes for $t = 0$ the Fourier expansion of the first term as an infinitely recurrent sawtooth wave with period $2s$, having its discontinuity (jump) at $x = 0$ and passing continuously through zero at $x = s$. The discontinuity at $x = 0$ implies very poor convergence of the Fourier series in its neighborhood and, in fact, the appearance of Gibbs' phenomenon* at $x = 0$ in the conventional summation; see also Vol. I, section 6. It is not possible to represent (131) as progressive waves similar to expansions (6.89) and (6.90) which we could apply for the double-energy transmission line; this re-emphasizes that the concept of propagation does not apply to the noninductive cable, but that it exhibits *diffusion* effects instead.

We might also consider the current (122) for the short-circuited cable and apply directly the theorem of residues. Introducing $\rho_t = 1$

* See particularly H. S. Carslaw, *Introduction to the Theory of Fourier's Series and Integrals*, 3rd edition, pp. 293-309, Dover Publications, New York, 1945.

and converting into the hyperbolic functions, (122) gives for an applied step source voltage

$$I(x, p) = \frac{V_m}{\sqrt{r/c}} \frac{\sqrt{p} \cosh(\sqrt{b} - \sqrt{a}) \sqrt{p}}{\sinh \sqrt{bp}} \frac{1}{p} \quad (132)$$

where we have used as before $a = rcx^2$, $b = rcs^2$, $n = \sqrt{ap}$. Again we can readily see that because of (127) the factor \sqrt{p} will cancel; the fraction is thus in fact a rational function and permits application of the partial fraction expansion. The root values are identical with (128) and we can apply directly (1.57), the conventional expansion theorem for the indicial admittance.

Defining

$$\frac{N(p)}{D(p)} = \frac{\sqrt{p} \cosh(\sqrt{b} - \sqrt{a}) \sqrt{p}}{\sinh \sqrt{bp}}$$

it reduces for $p = 0$ to $1/\sqrt{b} = (rcs^2)^{-1/2}$ and we have

$$\frac{N(p)}{p[dD(p)/dp]} = \frac{\sqrt{p} [\cosh \sqrt{bp} \cosh \sqrt{ap} - \sinh \sqrt{bp} \sinh \sqrt{ap}]}{p \sqrt{b} \frac{1}{2} \frac{1}{\sqrt{p}} \cosh \sqrt{bp}}$$

which simplifies for $p = p_\alpha$ from (128) to

$$\frac{2}{\sqrt{rc} s} \cos\left(\frac{x}{s} \alpha \pi\right)$$

If we now use the total resistance $R = rs$ and capacitance $C = cs$ for the line, we get for the current response

$$i(x, t) = \frac{V_m}{R} \left[1 + 2 \sum_{\alpha=1}^{\infty} \cos\left(\alpha \pi \frac{x}{s}\right) e^{-(\alpha \pi)^2 \frac{t}{RC}} \right] \quad (133)$$

This is certainly not a convergent series at $t = 0$ so that we must consider the whole process with caution. To verify the result, we might try to make use of relations (9) from the first of which we have for the noninductive cable

$$i(x, t) = -\frac{1}{r} \frac{\partial v(x, t)}{\partial x} \quad (134)$$

Using (131), we obtain the same result as (133); yet we should recognize that for $t = 0$ series (131) is not absolutely convergent and there-

fore may not be differentiated term by term! We can consider (131) as well as (133) valid as long as neither $x = 0$ nor $t = 0$.

If we are primarily interested in the receiving end current, we might choose as time scale

$$T = \frac{rcs^2}{\pi^2} = \frac{RC}{\pi^2}$$

and introduce the numerical time $\theta = t/T$, so that

$$i(s, \theta) = \frac{V_m}{R} \left[1 + 2 \sum_{\alpha=1}^{\infty} (-1)^\alpha e^{-\alpha^2 \theta} \right]$$

which represents a quickly convergent exponential series for $\theta > 0$; because it is alternating, we can easily determine the maximum error in accordance with section 7.3. A very similar result is obtained for the open-circuited cable.*

For more general lumped-parameter terminations we need to return to (121) and (122) together with (125). If we convert to hyperbolic functions again after first multiplying numerator and denominator by e^{ns} , we get for the Laplace transform of the current

$$\begin{aligned} I(x, p) &= \frac{\sqrt{r/c} \cosh n(s-x) + Z_t(p) \sqrt{p} \sinh n(s-x)}{\sqrt{r/c} \sinh ns + Z_t(p) \sqrt{p} \cosh ns} \frac{\mathcal{L}v_0(t)}{\sqrt{r/c} 1/\sqrt{p}} \\ &= \frac{\sqrt{r/c} \cosh (\sqrt{b} - \sqrt{a}) \sqrt{p} + Z_t(p) \sqrt{p} \sinh (\sqrt{b} - \sqrt{a}) \sqrt{p}}{\sqrt{r/c} \sqrt{p} \sinh \sqrt{bp} + Z_t(p) p \cosh \sqrt{bp}} \frac{p \mathcal{L}v_0(t)}{\sqrt{r/c}} \end{aligned}$$

Since $Z_t(p)$ must be a rational function of p for all lumped-parameter terminations, we see that $I(x, p)$ will again be a rational transcendental function of the *meromorphic* type so that we can apply the expansion into partial fractions, i.e., apply the theorem of residues directly. The poles are defined by the zeros of the denominator, namely

$$\frac{\sqrt{p} \sinh \sqrt{bp}}{\cosh \sqrt{bp}} = - \frac{p Z_t(p)}{\sqrt{r/c}} \quad (135)$$

* E. J. Berg, *Heaviside's Operational Calculus*, McGraw-Hill, New York, 1st edition, 1929, 2nd edition, 1936.

Realizing that for the noninductive cable all poles must be negative real as we have seen explicitly in (128), we might set

$$p_\alpha = -m_\alpha^2, \quad \sqrt{p_\alpha} = jm_\alpha$$

which converts the *characteristic equation* (135) into

$$\tan(m_\alpha \sqrt{b}) = -\frac{m_\alpha}{\sqrt{r/c}} Z_t(-m_\alpha^2) \quad (136)$$

This is now a real transcendental equation for the values m_α which defines the root values p_α . There is no explicit analytical method for extracting the roots of (136); the only feasible methods are graphical construction or numerical computation, or a combination of both for higher accuracy where this is warranted. One might, e.g., plot the left-hand side of (136) as the conventional tangent graph (or use directly tangent paper) and plot against $(m_\alpha \sqrt{b})$ also the right-hand side. The intersections give the first good approximations which can be improved by numerical computations. Obviously, the roots of (136) give the poles contributed by the meromorphic transmission-line function; to these must be added the poles contributed by $p\mathcal{L}v_0(t)$, if any.

PROBLEMS

7.1 A step voltage $V_0 I$ is applied at the sending end of an infinite ideal cable over the series resistance R similar to Fig. 7.6 where a capacitance is in series with the applied voltage. (a) Find the entering current of the cable. (b) Find the voltage along the cable by direct integration in the complex plane.

7.2 Give series developments in ascending and descending powers of p for the Laplace transform solutions in problem 7.1. (a) Identify the character of the series and convert them into the inverse time series. (b) For numerical computations assume $cR^2/r = 10^{-2}$ sec.

7.3 Take the same arrangement as in Fig. 7.6 and apply a sawtooth voltage pulse $v(t) = Vt/T$ for $t < T$, and $v(t) = 0$ for $t > T$. (a) Find the voltage response in terms of series developments in ascending and in descending powers of p . Identify the character of the series and convert into the corresponding time series if permissible. (b) Do the same for the current response.

7.4 Find the final steady current directly from the Laplace transform solution in problem 7.1. Verify this by physical reasoning.

7.5 Two square-wave voltage pulses each of duration T and with a time interval T between them are applied over the series resistance R as in problem 1 to the terminals of an infinitely long ideal cable. (a) Evaluate the entering current in response to these pulses in closed form; (b) plot the response for several values of T as related to $\gamma = cR^2/r$ and discuss the results.

7.6 An infinite sequence of square-wave voltage pulses is applied directly at the terminals of an infinitely long ideal cable. Find the eventual steady-state

response to this pulse sequence by at least two of the methods outlined in Vol. I, section 5.7.

7.7 Expand the Laplace transform solution (85) for the step voltage applied over the series inductance at the input end (shown in Fig. 7.9) into a series about $p \rightarrow \infty$. Convert into a time series; show its character and usefulness for small values of time.

7.8 Plot the response found in problem 7.7 for small values of time and also plot the response (89) for larger values of γt similar to Fig. 7.7, terminating the asymptotic series once with, and once above, the smallest term. Show the continuity of the two series solutions.

7.9 Evaluate the response of an infinitely long ideal cable to the voltage $v_0(t) = V_m \cos \omega t$. (a) Compute the voltage response for small values of time. (b) Compute the final steady-state voltage along the line.

7.10 Apply a sinusoidal voltage $V_m \sin \omega t$ to the ideal cable with series capacitance in Fig. 7.6. (a) Find the entering current to the line by series development in both ascending and descending powers of p , preserving the factor $(p^2 + \omega^2)^{-1}$ intact. (b) Interpret the individual terms by means of convolution integrals. (c) Use complex notation and evaluate its advantages.

7.11 A finite noninductive cable of length s is terminated at the far end into a resistance $R = \frac{1}{2}rs$. A step voltage $V_m I$ is applied at the sending end terminals. Evaluate the voltage distribution (a) by the exponential series expansion, (b) by the standing wave method.

7.12 A finite noninductive cable with leakance $g \neq 0$ is short-circuited at the far end. A sinusoidal voltage $V_m \sin \omega t$ is applied at the sending end terminals. Evaluate the voltage and current distributions by the standing wave method.

7.13 Take the same cable as in problem 7.12 but open-circuited at the far end. Evaluate the voltage distribution for a step voltage $V_m I$ applied at the sending end terminals.

7.14 A noninductive, leaky cable carries steady-state direct voltage and current with termination into $R = rs$ at both ends. At $t = 0$ the far end is suddenly short-circuited. Evaluate the subsequent current distribution along the cable, assuming that the power source remains connected at the sending end.

7.15 An ideal cable of short length is connected to an infinitely long lossless line. If a single rectangular pulse is applied at the input end of the cable, what will be observed on the lossless line?

7.16 A short distortionless line feeds an ideal cable (r, c only) of infinite length. If a single voltage pulse $V_m \cos \omega t$, $0 < \omega t < \pi$ is applied to the input terminals of the first line, what is the current at distance x from the beginning of the ideal cable?

7.17 An ideal cable of parameters $r = 0.5$ ohm/mile and $c = 0.05$ μ farad/mile and of length $s = 50$ miles is terminated into a series combination of $C = 10$ μ farads and $R = 1000$ ohms. Obtain (a) the standing wave solution for the current for applied unit step voltage, (b) the voltage across the terminals of condenser C for a single rectangular pulse of $T = 10^{-4}$ sec duration.

8. GENERAL TRANSMISSION LINES

As we can judge from Table 6.1, actual transmission lines or cables cannot readily be considered as either lossless or distortionless, even if practical loading has been introduced. This requires that attention be given to the more general treatment of transmission-line problems and in particular to the case of low but not negligible losses for power transmission lines, and to the case of negligible leakance for communication lines.

8.1 Transmission Lines With Low Losses

We return to the general statement of the Laplace transformed differential equations for transmission lines in section 6.3, restricting ourselves to the initially de-energized line. From (6.21) we take

$$\begin{aligned} -\frac{\partial V(x,p)}{\partial x} &= (r + pl)I(x,p) \\ -\frac{\partial I(x,p)}{\partial x} &= (g + pc)V(x,p) \end{aligned} \quad (1)$$

which we can combine for the voltage transform into the single, ordinary differential equation

$$\frac{d^2 V(x,p)}{dx^2} = n^2 V(x,p) \quad (2)$$

where we had defined in (6.24)

$$n^2 = (r + pl)(g + pc) = \frac{1}{u^2} [(p + \delta)^2 - \sigma^2] \quad (3)$$

Here, u is the velocity of propagation, δ the attenuation constant, and σ the distortion constant, defined respectively by

$$u = \frac{1}{\sqrt{lc}}, \quad \delta = \frac{1}{2} \left(\frac{r}{l} + \frac{g}{c} \right), \quad \sigma = \frac{1}{2} \left(\frac{r}{l} - \frac{g}{c} \right) \quad (4)$$

as repeated from (6.25).

Let us now assume that the losses are small, so that we can expand n from (3) binomially as follows

$$n = p \sqrt{lc} \left(1 + \frac{r}{pl}\right)^{1/2} \left(1 + \frac{g}{pc}\right)^{1/2} \approx \frac{p}{u} \left[1 + \frac{1}{2p} \left(\frac{r}{l} + \frac{g}{c}\right)\right]$$

To permit this expansion we must assume that

$$\frac{r}{|p|l} < 1, \quad \frac{g}{|p|c} < 1$$

where $|p|$ need only be the value along the path of integration for the inverse Laplace transform. If we shift this path in Fig. 1.11 far enough to the right of the imaginary axis, which we can always do for these passive transmission-line problems, then we can ensure that the preceding conditions are met. We thus have

$$n = \frac{1}{u} (p + \delta) = n_d \quad (5)$$

Essentially, then, for low losses the propagation function is identical with that for the distortionless line in (6.35), assuring the same simple treatment of propagation problems as there. The characteristic impedance from (6.30), on the other hand, by the same expansion takes the form

$$\begin{aligned} Z_c &= \sqrt{\frac{l}{c}} \left(1 + \frac{r}{pc}\right)^{1/2} \left(1 + \frac{g}{pc}\right)^{-1/2} \approx \sqrt{\frac{l}{c}} \left[1 + \frac{1}{2p} \left(\frac{r}{l} - \frac{g}{c}\right)\right] \\ Z_c &= R_s \left(1 + \frac{\sigma}{p}\right) \end{aligned} \quad (6)$$

which is remarkably different from the distortionless case, yet simple enough to encourage complete solutions. We can interpret this physically as the series combination of the surge resistance, characteristic for both lossless and distortionless transmission, and a fictitious capacitance C_s independent of the length of the line

$$Z_c = R_s + \frac{1}{pC_s} \quad (7)$$

where C_s is given by

$$C_s = (\sigma R_s)^{-1} = c \frac{u}{\sigma} = \frac{2c}{\frac{r}{R_s} - gR_s} \quad (8)$$

This will introduce definite distortion in the transient response more in accord with observed data.*

Suppose we apply directly to the infinitely long line a signal voltage $v_0(t)$. The transform solution is the same as (6.44),

$$V(x,p) = B e^{-\frac{\delta x}{u}} e^{-p \frac{x}{u}}, \quad I(x,p) = \frac{V(x,p)}{Z_c} \quad (9)$$

giving the outgoing traveling wave. At the line terminals we have as the sending end boundary condition

$$V(0,p) = \mathcal{L}v_0(t) = B$$

which gives at once the only available integration constant. The voltage wave is therefore exactly the same as for the distortionless line in (6.45)

$$v(x,t) = e^{-\frac{\delta x}{u}} v_0 \left(t - \frac{x}{u} \right) \quad t \geq \frac{x}{u} \quad (10)$$

The current wave, however, will now not have the same shape as this voltage wave! Taking Z_c from (6), we find

$$I(x,p) = \frac{p}{R_s(p + \sigma)} e^{-\frac{\delta x}{u}} \mathcal{L}v_0(t) e^{-p \frac{x}{u}} \quad (11)$$

The inverse Laplace transform can be best obtained by separating

$$\frac{p}{p + \sigma} = 1 - \frac{\sigma}{p + \sigma}$$

The first part gives identically the distortionless response, the second part can be identified by means of the convolution integral line 11 in Table 1.5. The result is then

$$i(x,t) = \frac{e^{-\frac{\delta x}{u}}}{R_s} v_0 \left(t - \frac{x}{u} \right) - \sigma \frac{e^{-\frac{\delta x}{u}}}{R_s} e^{-\sigma t} \int_{t=x/u}^t v_0 \left(t - \frac{x}{u} \right) e^{\sigma t} dt \quad (12)$$

As $\sigma \rightarrow 0$ we approach the solution for the distortionless line in (6.45) as it should be. However, in general there will be a noticeable distortion of the signal current.

* R. Pélissier, "Propagation of Transient and Periodic Waves Along Transmission Lines" [French], *Rev. gén. élec.*, **59**, 379-399, 437-454, 502-512 (1950).

To have a definite comparison, we might specify a simple rectangular pulse

$$v_0(t) = V_m[S_{-1}(t) - S_{-1}(t - \tau)]$$

if τ is the pulse duration. The Laplace transform is

$$V_0(p) = \frac{V_m}{p} (1 - e^{-p\tau})$$

which we introduce best directly into (11)

$$I(x, p) = \frac{V_m}{R_s} e^{-\delta \frac{x}{u}} \frac{e^{-p \frac{x}{u}}}{p + \sigma} (1 - e^{-p\tau})$$

The inverse transform of the simple rational fraction is found in Table 1.3, line 4; the exponential $e^{-\delta \frac{x}{u}}$ means local delay caused by the finite velocity of propagation u , and $e^{-p\tau}$ is the superposition of negative unit step at time $t = \tau$ to shape the pulse. Thus

$$i(x, t) = \frac{V_m}{R_s} e^{-\delta \frac{x}{u}} \left[e^{-\sigma \left(t - \frac{x}{u} \right)} S_{-1} \left(t - \frac{x}{u} \right) - e^{-\sigma \left(t - \tau - \frac{x}{u} \right)} S_{-1} \left(t - \tau - \frac{x}{u} \right) \right] \quad (13)$$

The current pulse is therefore exactly like the response of a simple R - C circuit to the rectangular voltage pulse as shown in Vol. I, Fig. 5.9, for the first period T . For a periodic succession of pulses, Fig. 5.9 would give the identical response at any fixed x , with time origin defined by $t = x/u$, the arrival of the first pulse front, and with a vertical scale exponentially shrunk as indicated by $e^{-\delta(u/x)}$. It is important to observe that the *attenuation along the line* is given by δ as one would expect, but the *local time constant* for the current response is $1/\sigma$.

For the finite line with low losses, we can readily expand the complete solution into traveling waves exactly as we did in (6.51) and (6.52) for the lossless line, making the same adjustment of p to $p + \delta$ in the exponential terms and taking Z_c from (6). For the Laplace transform of the voltage we thus have

$$V(x, p) = \frac{Z_c}{Z_c + Z_0} \mathcal{L}v_0(t) \left(e^{-\delta \frac{x}{u}} e^{-p \frac{x}{u}} - \rho_t e^{-\delta \frac{2s-x}{u}} e^{-p \frac{2s-x}{u}} \right. \\ \left. + \rho_0 \rho_t e^{-\delta \frac{2s+x}{u}} e^{-p \frac{2s+x}{u}} - \dots \right)$$

For the current transform we divide by Z_c (omit it in the first numerator) and change all negative signs to positive ones. The reflection

factor at the receiving end $x = s$ becomes

$$\rho_t = \frac{Z_c - Z_t}{Z_c + Z_t} = \frac{R_s - Z_t + R_s \frac{\sigma}{p}}{R_s + Z_t + R_s \frac{\sigma}{p}}$$

where the first terms in numerator and denominator are the same ones as for the distortionless line. In fact, we might approximate to first order in p^{-1}

$$\rho_t \approx \frac{R_s - Z_t}{R_s + Z_t} \left(1 + \frac{\sigma}{p} R_s \frac{2Z_t}{R_s^2 - Z_t^2} \right) \quad (14)$$

to bring out the correction to the distortionless case. If we take the same example as in section 6.6, $Z_0 = Z_t = \frac{1}{2}R_s$, we obtain

$$\rho_0 = \rho_t = \frac{1}{3} \left(1 + \frac{4}{3} \frac{\sigma}{p} \right)$$

Let us now apply a step voltage $V_m 1$. The first outgoing voltage wave will be identical with solution (6.55) for the distortionless case

$$v_1(x, t) = \frac{2V_m}{3} e^{-\delta \frac{x}{u}} S_{-1} \left(t - \frac{x}{u} \right)$$

The second wave, or first reflected wave, in the distortionless line was a replica of v_1 but of relative magnitude $\frac{1}{3}$ because of $\rho_t = \frac{1}{3}$ and the appropriate further attenuation. For the line with nonnegligible losses we now get for the Laplace transform of the second wave

$$V_2(x, p) = -\frac{2}{9} \frac{V_m}{p} e^{-\delta(2s-x)/u} e^{-p(2s-x)/u} \left(1 + \frac{4}{3} \frac{\sigma}{p} \right)$$

We thus have the first part of the inverse Laplace transform identical with (6.56), but we need to add another term

$$-\frac{8}{27} \sigma V_m \left(t - \frac{2s-x}{u} \right) e^{-\delta(2s-x)/u} S_{-1} \left(t - \frac{2s-x}{u} \right)$$

which makes the total voltage wave $v_2(x, t)$ appear linearly decreasing from the initial value which remains the same as in the distortionless line. Obviously, the third term from (6.51), the reflected wave at the sending end upon arrival of $v_2(x, t)$ at $x = 0$, will have the additional factor

$$\rho_0(p) = \frac{1}{3} + \frac{4}{9} \frac{\sigma}{p}$$

The first term simply reproduces $v_2(x,t)$ with reduced magnitude; the second factor constitutes an additional distortion term which can easily be interpreted as integration in time by line 4a of Table 1.4. This results now in a parabolic variation in time.

The successive reflections thus produce increased distortion, explaining the experimental results from oscillograms of traveling waves on lines in rather simple fashion.* In the same reference, the influences of skin effect and corona are also studied by introducing a term $2\eta l \sqrt{p}$ added to the resistance r of the line, where η is a constant and l the inductance per unit length. This results then with the same degree of approximation as in (5) in the final form

$$n = \frac{1}{u} [(p + \delta) + \eta \sqrt{p}]$$

leading to solutions quite similar to those we found in the ideal cable, though now with genuine propagation effect. For the infinite line, the voltage transform from (9) is modified into

$$V(x,p) = B e^{-\frac{\delta x}{u} e^{-p \frac{x}{u}} - \frac{\eta x}{u} \sqrt{p}} \tag{15}$$

for which we can readily find the inverse transform. Assuming again a step voltage $V_m 1$ applied directly to the terminals of the line so that $B = V_m/p$, we first evaluate the inverse transform of

$$\frac{V_m}{p} e^{-\frac{\eta x}{u} \sqrt{p}}$$

This is the same problem as (7.10) for the ideal cable and has the solution (7.45), namely

$$V_m \operatorname{erfc} \left(\frac{\eta}{u} \frac{x}{2 \sqrt{t}} \right)$$

represented graphically in Fig. 7.4. The complete inverse transform to (15) is then

$$v(x,t) = V_m e^{-\frac{\delta x}{u}} \operatorname{erfc} \left(\frac{\eta}{u} \frac{x}{2 \sqrt{t - (x/u)}} \right) \quad t \geq \frac{x}{u} \tag{16}$$

We have attenuation as in the distortionless line, and direct propagation of the shape of Fig. 7.4 along the line. The wave front of the abrupt step has been transformed into a gradual rise, as borne out by experimental evidence.

* Pélissier, *op. cit.*, pp. 393–396.

The solution for the current will be more involved because of the characteristic impedance Z_c which is also modified by the influence of the skin effect and can be approximated by

$$Z_c = R_s \left(1 + \frac{\sigma}{p} + \frac{\eta}{\sqrt{p}} \right)$$

8.2 The Infinitely Long General Transmission Line

For the general transmission line it is necessary to solve the transform differential equations (1) in full generality. This requires the complete solution for either voltage or current transform of the second-order differential equation

$$\frac{d^2 F(x, p)}{dx^2} = n^2 F(x, p) \quad (17)$$

with n^2 defined by (3) without further simplifications. For convenience, we choose first the infinitely long line and apply a step voltage to it directly across the input terminals. The general solution of (17) for the voltage transform gives as in (6.27)

$$V(x, p) = Ae^{nx} + Be^{-nx} \quad (18)$$

and for the current correspondingly as in (6.29)

$$I(x, p) = \frac{1}{Z_c} (-Ae^{nx} + Be^{-nx}) \quad (19)$$

We still will write the propagation function $n(p)$ as in (3)

$$n^2 = (r + pl)(g + pc) = \frac{1}{u^2} [(p + \delta)^2 - \sigma^2] \quad (20)$$

presuming that the essential meanings of u , δ , and σ remain the same as originally explained in section 6.3, following (6.32). The characteristic impedance will have the general form (6.30)

$$Z_c = \sqrt{\frac{r + pl}{g + pc}} = \sqrt{\frac{1}{c}} \sqrt{\frac{p + \delta'}{p + \delta''}} = \frac{n}{g + pc} \quad (21)$$

where

$$\delta' = \frac{r}{l} = \delta + \sigma, \quad \delta'' = \frac{g}{c} = \delta - \sigma \quad (22)$$

We assume that n has a positive real part

$$\operatorname{Re}(n) = \operatorname{Re} \frac{1}{u} [(p + \delta)^2 - \sigma^2]^{\frac{1}{2}} > 0 \quad (23)$$

i.e., we select the positive root so that this is the case. Then we have assurance that the solution for the infinite line must discard the first terms in (18) and (19). The boundary condition at $x = 0$ is satisfied by identifying $V(0, p)$ with the Laplace transform of the applied voltage, which we chose as $v_0(t) = V_m t$

$$V(0, p) = B = \mathfrak{L}v_0(t) = \frac{V_m}{p}$$

We thus have the final solutions for the respective Laplace transforms

$$V(x, p) = \frac{V_m}{p} \exp \left(-\frac{x}{u} \sqrt{(p + \delta)^2 - \sigma^2} \right) \quad (24)$$

and

$$I(x, p) = u \frac{V_m}{p} (g + pc) \frac{\exp \left(-\frac{x}{u} \sqrt{(p + \delta)^2 - \sigma^2} \right)}{\sqrt{(p + \delta)^2 - \sigma^2}} \quad (25)$$

where we used the last form of Z_c in (21).

Though (24) looks like the simpler form, we actually can find the inverse Laplace transform more readily for the last factor in (25) as we will show in section 8.3. Indeed, we have from pair 3 in Table 8.1

$$\mathfrak{L}^{-1} \frac{1}{Q} e^{-\frac{x}{u} Q} = e^{-\delta t} I_0 \left[\sqrt{t^2 - \left(\frac{x}{u} \right)^2} \right] = W_x(t) \quad t > \frac{x}{u} \quad (26)$$

$$Q = [(p + \delta)^2 - \sigma^2]^{\frac{1}{2}}$$

$I_0(y)$ is the *modified Bessel function* of first kind and zero'th order, related to the ordinary Bessel function by

$$J_0(jy) = I_0(y)$$

This relationship is very similar to that existing between trigonometric and hyperbolic functions, as, e.g.

$$\frac{d}{dy} J_0(y) = -J_1(y) \quad \frac{d}{dy} \cos y = -\sin y$$

$$J_0(jy) = I_0(y) \quad \cos jy = \cosh y$$

$$J_1(jy) = jI_1(y) \quad \sin jy = j \sinh y$$

TABLE 8.1
LAPLACE TRANSFORMS OF BESSEL FUNCTIONS (WAVE PROPAGATION)

No.	Time Function	Laplace Transform	Notes (Reference Eq.)
1	$J_0(\alpha \sqrt{t^2 - (x/u)^2})$	$\frac{\exp(-\theta \sqrt{p^2 + \alpha^2})}{\sqrt{p^2 + \alpha^2}}$	Same as 2 with $\alpha = j\sigma, t > x/u$
2	$I_0[\sigma \sqrt{t^2 - (x/u)^2}]$	$\frac{\exp(-x/u \sqrt{p^2 - \sigma^2})}{\sqrt{p^2 - \sigma^2}}$	Same as 3 with $\delta = 0, t > x/u$
3	$e^{-\delta t} I_0[\sigma \sqrt{t^2 - (x/u)^2}]$	$\frac{\exp[-x/u \sqrt{(p + \delta)^2 - \sigma^2}]}{\sqrt{(p + \delta)^2 - \sigma^2}}$	$t > x/u$, (44) and (41)
4	$e^{-\frac{\alpha}{2}t} I_0[(\alpha/2) \sqrt{t^2 - (x/u)^2}]$	$\frac{\exp[-x/u \sqrt{p(p + \alpha)}]}{\sqrt{p(p + \alpha)}}$	Same as 3 with $\delta + \sigma = \alpha$ and $\delta - \sigma = 0$, where $t > 0$
5	$e^{-\frac{x}{u}t} S_0\left(t - \frac{x}{u}\right) + \sigma \frac{u}{x} \frac{e^{-\delta t}}{\sqrt{t^2 - (x/u)^2}} I_0[\sigma \sqrt{t^2 - (x/u)^2}]$	$\exp\left[-\frac{x}{u} \sqrt{(p + \delta)^2 - \sigma^2}\right]$	Time derivative of 6, $t \geq x/u$
6	$e^{-\frac{x}{u}t} S_{-1}\left(t - \frac{x}{u}\right) + \sigma \frac{x}{u} \int_{t=\frac{x}{u}}^t \frac{e^{-\delta t}}{\sqrt{t^2 - (x/u)^2}} I_0[\sigma \sqrt{t^2 - (x/u)^2}] dt$	$\frac{1}{p} \exp\left[-\frac{x}{u} \sqrt{(p + \delta)^2 - \sigma^2}\right]$	$t \geq x/u$, (31)

Related Functions:

11	$J_0(\alpha t)$	$\frac{1}{\sqrt{p^2 + \alpha^2}}$	Table 3.1, no. 2 (33)
12	$I_0(\sigma t)$	$\frac{1}{\sqrt{p^2 - \sigma^2}}$	Table 3.1, no. 12 (39)
13	$J_0(\alpha \sqrt{t})$	$\exp\left(-\frac{\alpha^2}{4p}\right)$	(40)
14	$I_0(\sigma \sqrt{t})$	$\exp\left(+\frac{\sigma^2}{4p}\right)$	Same as 13 with $\alpha = j\sigma$
15	$e^{-\delta t} J_0(\alpha t)$	$\frac{1}{\sqrt{(p + \delta)^2 + \alpha^2}}$	(11) with (23)
16	$e^{-\delta t} I_0(\sigma t)$	$\frac{1}{\sqrt{(p + \delta)^2 - \sigma^2}}$	Same as 15 with $\alpha = j\sigma$
17	$e^{-\frac{\alpha}{2}t} I_0\left(\frac{\alpha t}{2}\right)$	$\frac{1}{\sqrt{p} \sqrt{p + \alpha}}$	Same as 16 with $\delta + \sigma = \alpha$, $\delta - \sigma = 0$

General Formal Relations:

21	$S_{-1}\left(t - \frac{x}{u}\right)$	$\frac{1}{p} e^{-p \frac{x}{u}}$	$\frac{1}{p} = \mathcal{L}S_{-1}(t)$
22	$f\left(t - \frac{x}{u}\right)$	$F(p) e^{-p \frac{x}{u}}$	$F(p) = \mathcal{L}f(t)$
23	$e^{-\delta t} f(t)$	$F(p + \delta)$	$F(p) = \mathcal{L}f(t)$

This is borne out also by comparing the power series developments given in section 8.3. We thus expect that the time function (26) will start from value $e^{-\delta t}$ and either increase to a maximum and then decline to final value zero as $t \rightarrow \infty$ or directly decrease from its initial value depending upon the relative values of δ and σ . To demonstrate the behavior for $t \rightarrow \infty$ we utilize the asymptotic development of the modified Bessel function*

$$\lim_{t \rightarrow \infty} I_0 \left[\sigma \sqrt{t^2 - \left(\frac{x}{u}\right)^2} \right] = \lim_{t \rightarrow \infty} I_0(\sigma t) = \frac{e^{\delta t}}{\sqrt{2\pi\sigma t}} \tag{27}$$

Introducing this into (26), mindful of the definitions of δ and σ in (4), we obtain

$$\lim_{t \rightarrow \infty} W_x(t) = \frac{e^{(-\delta + \sigma)t}}{\sqrt{2\pi\sigma t}} = \frac{e^{-\frac{g}{c}t}}{\sqrt{2\pi\sigma t}} \rightarrow 0$$

Even if we should have $g = 0$, i.e., no leakance, the asymptotic behavior is defined by $t^{-1/2}$.

The complete solution for the current can now be obtained from (25) with (26) and interpreting the second term in

$$\frac{u}{p}(g + pc) = \sqrt{\frac{c}{l}} + \frac{ug}{p}$$

by Table 1.4 as definite integration. The result is

$$i(x,t) = \frac{V_m}{\sqrt{l/c}} W_x(t) + ugV_m \int_{t=x/u}^t W_x(t) dt \qquad t \geq \frac{x}{u} \tag{28}$$

The second term is directly proportional to the leakance g which is normally quite small so that for most practical applications we can disregard this integral and write in good approximation

$$i(x,t) \approx \frac{V_m}{R_s} e^{-\delta t} I_0 \left[\sigma \sqrt{t^2 - \left(\frac{x}{u}\right)^2} \right] \qquad t \geq \frac{x}{u} \tag{29}$$

An oscillogram at a point x_1 will then show at time $t_1 = x_1/u$ a sudden rise of current to the value

$$i(x_1,t_1) = \frac{V_m}{R_s} e^{-\delta \frac{x_1}{u}}$$

* T. Kármán and M. Biot, *Mathematical Methods in Engineering*, chapter II, particularly p. 61, McGraw-Hill, New York, 1940. Also L. Weber, *Electromagnetic Fields*, Vol. I—*Mapping of Fields*, Appendix 5, Wiley, New York, 1950.

which itself decreases exponentially along the line. Fig. 8.1. shows this exponential $e^{-\delta t}$ and three current curves computed respectively at $x = 0$, the terminals of the line, at an arbitrary distance x , and at twice that distance. For these computations we assumed $\sigma = \delta$ or $g = 0$ for simplicity. It is evident that for $\delta(x_1/u) < 2$ the exponential function predominates in (29), and that for $\delta(x_1/u) > 2$ the Bessel function dictates a current rise above the initial surge front. For $\delta(x_1/u) = 2$ we have nearly constant value up to $\delta(x_1/u) \cong 4$ and then a slow decline. Tables for the function $I_0(y) = J_0(jy)$ are given in

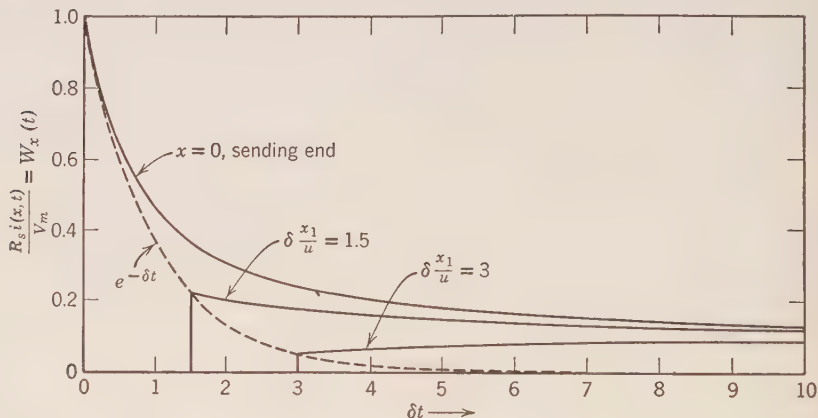


Fig. 8.1. Current response to a step voltage of a transmission line with $g = 0$, or $\sigma = \delta$.

Jahnke-Emde and McLachlan.* Because $\sigma = \delta$ is the largest value that σ can attain, as seen from definition (4), in cases where $\sigma < \delta$ the current response will show a quicker decline than given in Fig. 8.1.

Returning now to the voltage transform solution (24), we observe that, using the notation of (26)

$$\frac{d}{dx} e^{-\frac{x}{u}Q} = -\frac{Q}{u} e^{-\frac{x}{u}Q}$$

This differentiation is quite permissible because the exponential is a continuous function of x . This permits us now to interpret the inverse Laplace transform of (24) as the time integral

$$v(x,t) = V_m \int_0^t -u\mathcal{E}^{-1} \frac{d}{dx} \left(\frac{1}{Q} e^{-\frac{x}{u}Q} \right) dt \quad (30)$$

* E. Jahnke and F. Emde, *Tables of Functions*, 3rd edition, pp. 226-232, Dover Publications, New York, 1943. N. W. McLachlan, *Bessel Functions for Engineers*, Oxford University Press, England, 1934.

With proper care, we can interchange the differentiation $d/(dx)$ and the inversion of the transform, i.e., we can differentiate $W_x(t)$ from (26) with respect to x . The curves in Fig. 8.1 show a sudden rise at the local arrival time of the current wave. The derivative of (26) must therefore show an impulse function at $t = x/u$, so that we obtain

$$\left\{ e^{-\delta t} \frac{d}{dx} I_0 \left[\sigma \sqrt{t^2 - \left(\frac{x}{u} \right)^2} \right] \right\}_{t > x/u} + [e^{-\delta t} S_0(t)]_{t=x/u}$$

The derivative of the Bessel function leads to

$$\frac{d}{dx} I_0 \left[\sigma \sqrt{t^2 - \left(\frac{x}{u} \right)^2} \right] = -I_1 \left[\sigma \sqrt{t^2 - \left(\frac{x}{u} \right)^2} \right] \frac{-\sigma(x/u)}{\sqrt{t^2 - (x/u)^2}}$$

This must be introduced into the time integral (30). We observe that the term containing the unit impulse becomes

$$e^{-\delta \frac{x}{u}} S_{-1} \left(t - \frac{x}{u} \right)$$

because the unit impulse occurs at $t = x/u$. The Bessel function is subject to the same restriction as $W_x(t)$ in (26); it has zero value for $t < x/u$ so that the lower limit in (30) should be replaced by x/u . The total result is thus

$$v(x, t) = V_m e^{-\delta \frac{x}{u}} S_{-1} \left(t - \frac{x}{u} \right) + \sigma V_m \frac{x}{u} \int_{t=x/u}^t \frac{e^{-\delta t}}{\sqrt{t^2 - (x/u)^2}} I_1 \left[\sigma \sqrt{t^2 - \left(\frac{x}{u} \right)^2} \right] dt \quad (31)$$

The physical interpretation of this solution is interesting. The first term is quite obviously the identical solution for the distortionless line given in (6.45) and found again for the line with low losses in (10). The second term explicitly describes the effect of $\sigma \neq 0$. The voltage rises above the value of the initial wave front and has a final value that is defined by the steady-state leakance along the line. If $g = 0$, then the voltage must ultimately reach the uniform value V_m all along the line. We can readily verify this by using the general relation 1b from Table 1.4, namely

$$v(x, \infty) = \lim_{p \rightarrow 0} pV(x, p) = V_m e^{-x/u \sqrt{\delta^2 - \sigma^2}} = V_m \exp \left(-\frac{x}{u} \sqrt{\frac{r}{l} \frac{g}{c}} \right) \quad (32)$$

For $g \neq 0$, we have an exponential steady-state distribution of the voltage along the line; for $g = 0$, we have, $v(x, \infty) = V_m$.

The exact evaluation of the integral in (31) can only be done graphically or numerically. Approximate forms of the voltage build-up are indicated for $g = 0$ in Fig. 8.2 for the same points along the line as the current responses in Fig. 8.1. It is impressive to observe the very slow build-up of the voltage at distant points along the line. Similar graphs are given for different conditions in Carson,* who also was the first to give the complete solutions (28) and (31) though by a different method.† The indicial admittance for the line without leakage in (29) had been given by Heaviside.‡

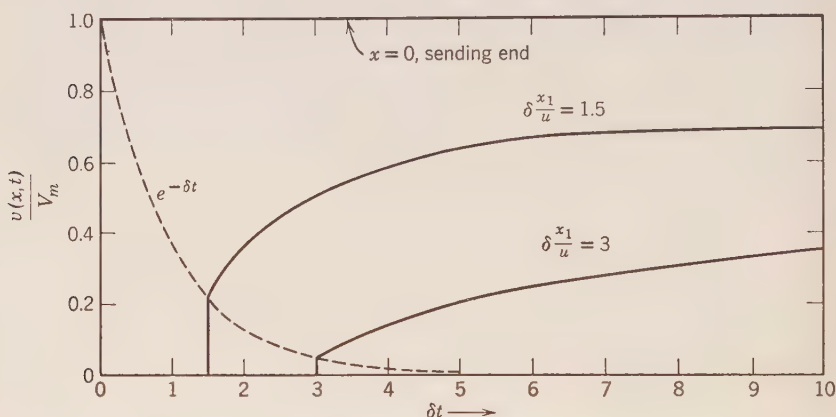


Fig. 8.2. Voltage build-up along a transmission line with $g = 0$, or $\sigma = \delta$ in response to a step voltage applied at the sending end.

Based upon the solutions for the unit step response, we can find the solutions to other voltage or current forms by means of the superposition theorems in Table 1.5. However, most of the necessary integrations cannot be performed in closed form, underlining the advantages gained by the simplifying assumptions treated in the previous sections.

8.3 Evaluation of Contour Integrals Involving Bessel Functions

We demonstrated in section 3.7 that the Laplace transform of the simple Bessel functions of the first kind could be obtained as a solution of the Laplace transformed differential equation of the Bessel functions.

* J. R. Carson, *Electric Circuit Theory and Operational Calculus*, chapter VII, McGraw-Hill, New York, 1926.

† J. R. Carson, "Theory of the Transient Oscillations of Electrical Networks and Transmission Lines," *Trans. AIEE*, **38**, 345-427 (1919).

‡ Oliver Heaviside, *Electromagnetic Theory*, Vol. I and II, E. Benn, London, 1893; also Dover Publications, New York, 1950.

We take as a result from Table 3.1, line 2

$$\mathfrak{L}J_0(\sigma t) = \frac{1}{\sqrt{p^2 + \sigma^2}}$$

or reversely

$$\frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^{pt} \frac{dp}{\sqrt{p^2 + \sigma^2}} = J_0(\sigma t) \tag{33}$$

defining the zero'th-order Bessel function in terms of the inverse Laplace transform integral. The integrand has two branch points located at $p = \pm j\sigma$ as shown in Fig. 8.3. For the evaluation of the

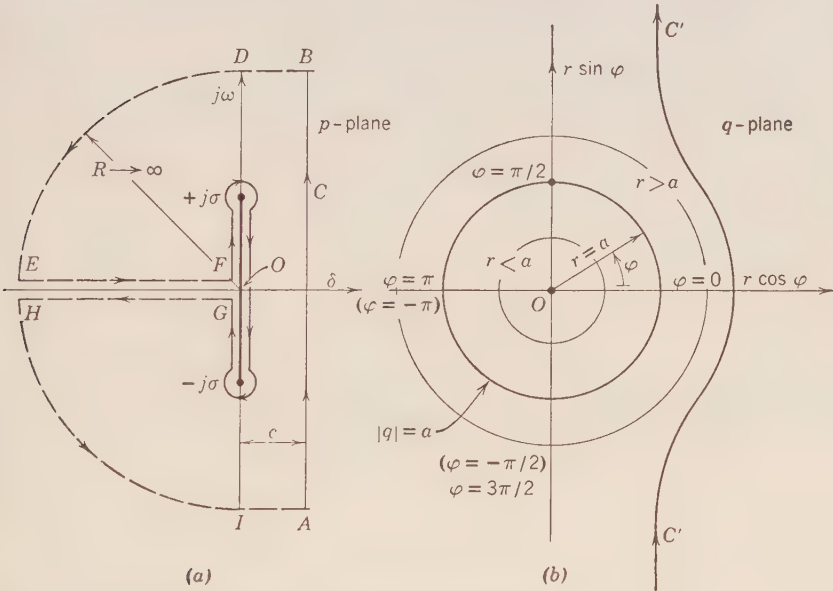


Fig. 8.3. Path for the inverse Laplace transform leading to $J_0(\sigma t)$. (a) p -plane, branch cut extends from $-j\sigma$ to $+j\sigma$; (b) q -plane, circle $|q| = a$ is conformal map of the branch cut.

integral we utilize Cauchy's integral theorem as we have done in the review section 1.4 as well as in section 7.1 for the ideal cable solution involving a single branch point. We can complete the regular path C of the inverse Laplace transform by the dotted boundary over the quarter circles DE and HI and the parallel lines EF and GH , leading to the contour around the branch cut. The entire closed line integral gives zero value because no other singularity exists except the two branch points which have been excluded. We have shown in Vol. I,

section 5.8, that the contributions over the very large circular arcs vanish for any proper fraction of rational functions. If we let $|p|$ be very large in the integrand of (33) we see that it approaches the form of the inverse transform of unit step so that we can make the same supposition. Actually, the two parallel paths EF and GH lie in an entirely regular region of the integrand so that their combined contribution must also be zero. This means then that we can replace the conventional path C for the inverse Laplace transform by the contour around the branch cut but in reverse direction to that indicated, since we want to call it an equivalent path to C . Other equivalent paths exist as shown by McLachlan,^{D13} p. 57.

We can transform the integral (33) by the change of variable

$$q = \frac{1}{2}t(p + \sqrt{p^2 + \sigma^2}) \quad (34)$$

where t is a real parameter, usually time in our own applications, and q a new complex variable with the derivative

$$dq = \frac{1}{2}t \left(1 + \frac{p}{\sqrt{p^2 + \sigma^2}} \right) dp = \frac{t}{2\sqrt{p^2 + \sigma^2}} (p + \sqrt{p^2 + \sigma^2}) dp$$

or also with (34)

$$\frac{dq}{q} = \frac{dp}{\sqrt{p^2 + \sigma^2}}$$

The exponent in (33) becomes from (34), upon separating the square root and squaring both sides

$$pt = q - \frac{\sigma^2 t^2}{4q} \quad (35)$$

The limits can remain the same, because for p very large

$$\lim_{p \rightarrow \infty} q = \frac{1}{2}t(p + p) = pt$$

and since t is real, we have essential identity except for scale.

The equivalent integral is thus

$$\frac{1}{2\pi j} \int_{m-j\infty}^{m+j\infty} \exp\left(q - \frac{\sigma^2 t^2}{4q}\right) \frac{dq}{q} = J_0(\sigma t) \quad (36)$$

with m a real positive value. This integral is not of the Laplace type; it is to be performed in the q -plane along a path roughly the equivalent of C in Fig. 8.3. In the inverse Laplace integral we had assumed $\text{Re } p > 0$ and the same will be true of the corresponding path in the

q-plane. As we see directly from (34), we will have $\text{Re } q > 0$. This is important because the new integral (36) has a pole and an essential singularity at $q = 0$.

To be more accurate, we need to consider the conformal transformation* from the *p*-plane to the *q*-plane defined by (34) and (35). If we write *q* in polar form

$$q = re^{j\varphi} = r \cos \varphi + jr \sin \varphi$$

and substitute for $pt = w$ for simplicity of writing, then (35) gives

$$w = u + jv = r(\cos \varphi + j \sin \varphi) - \left(\frac{\sigma t}{2}\right)^2 \frac{1}{r} (\cos \varphi - j \sin \varphi)$$

Separating real and imaginary parts and using $\sigma t/2 = a$, we obtain

$$\begin{aligned} u &= \left(r - \frac{a^2}{r}\right) \cos \varphi \\ v &= \left(r + \frac{a^2}{r}\right) \sin \varphi \end{aligned} \tag{37}$$

Thus *w* (*q*) transforms circles in the *q* plane into ellipses in the *w* = *pt* plane, as we see by eliminating φ in (37),

$$\left(\frac{u}{r - (a^2/r)}\right)^2 + \left(\frac{v}{r + (a^2/r)}\right)^2 = 1$$

The major axes *m_a* of the ellipses are along the *v*-axis in the *w* = *pt* plane, the minor axes *m_b* are along the *u*-axis, of magnitudes

$$m_a = r + \frac{a^2}{r}, \qquad m_b = r - \frac{a^2}{r}$$

and the distances of the foci from the origin are

$$m_f = \pm \sqrt{m_a^2 - m_b^2} = \pm 2a = \pm \sigma t$$

independent of the radius of the circle in the *q*-plane. Thus, the ellipses are all confocal, and the two branch points in the *p*-plane $\pm j\sigma$ become the foci $\pm j\sigma t$ in the *w* = *pt* plane.

The circle $r = a$ is a singular one, having the branch cut as its conformal map in the *w*-plane. All the points inside the circle $|q| = a$ fill one leaf of the *w*-plane. All the points outside the circle $|q| = a$ fill the second leaf of the *w*-plane, and both leaves are connected *only* along the branch cut. Thus the point $w = pt \rightarrow \infty$ remains separated in the two leaves, substantiating that we can retain the same limits in the

* E. Weber, *op. cit.*, pp. 319-320.

transformed contour integral (36). To relate points along the circle $|q| = a$ to corresponding points in the w -plane, we need to also eliminate r from (37) which leads to

$$-\left(\frac{u}{\cos \varphi}\right)^2 + \left(\frac{v}{\sin \varphi}\right)^2 = (2a)^2$$

or, if we prefer

$$\left(\frac{v}{2a \sin \varphi}\right)^2 - \left(\frac{u}{2a \cos \varphi}\right)^2 = 1$$

the normal form of the orthogonal family of hyperbolas. We can thus follow through the variation of $0 < \varphi < 2\pi$ along the branch cut for which $u = 0$, so that

$$v = 2a \sin \varphi, \quad u = 0$$

The branch cut appears exactly as the compressed circle of the q -plane with $\varphi = \pm(\pi/2)$ corresponding to the end point of the branch cut and with $\varphi = 0, \pm\pi$ being the opposite point at the origin O .

To judge the path of integration in the q -plane we observe that we have committed ourselves to reach $m \pm j\infty$ in (36). This can be done only if we select the branch $|q| > a$ and thus path C' in the z -plane which can be computed from (34), inserting $p = c + j\omega$. Now we could contract this path into the circle $|q| = a$, remaining on the *outer rim* of it, because there is no singularity of the integrand (36) at $|q| > a$. If we write along this circle

$$q = ae^{j\varphi}, \quad dq = jae^{j\varphi} d\varphi$$

we can transform the integral (36) further with

$$q - \frac{a^2}{q} = ae^{j\varphi} - ae^{-j\varphi} = 2aj \sin \varphi = j\sigma t \sin \varphi$$

into the form

$$\frac{1}{2\pi} \int_{\varphi=0}^{2\pi} \exp(j\sigma t \sin \varphi) d\varphi = J_0(\sigma t) \quad (38)$$

which is a standard integral definition treated in advanced books on Bessel functions.* Similarly, integral relations can be deduced for higher order Bessel functions, as well as for the different kinds.†

* R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. I, chapter VII, section 2, Interscience, New York, 1953; G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, England, 1922.

† N. W. McLachlan, *loc. cit.*; also E. A. Guillemin, *The Mathematics of Circuit Analysis*, chapter VII, article 16, Wiley, New York, 1949.

Returning to (33) and (36), we see at once that a change from σ to $j\sigma$ produces very simple changes in the integrands, but leads to the modified Bessel function of the first kind

$$\begin{aligned} J_0(j\sigma t) &= I_0(\sigma t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^{pt} \frac{dp}{\sqrt{p^2 - \sigma^2}} \\ &= \frac{1}{2\pi j} \int_{m-j\infty}^{m+j\infty} \exp\left(q + \frac{\sigma^2 t^2}{4q}\right) \frac{dq}{q} \end{aligned} \quad (39)$$

The contour integral (36) is a particularly good form to introduce various arguments of the Bessel function. Thus, we see readily that

$$J_0(\sigma \sqrt{t}) = \frac{1}{2\pi j} \int_{m-j\infty}^{m+j\infty} \exp\left(q - \frac{\sigma^2 t}{4q}\right) \frac{dq}{q}$$

This integral can be converted into the Laplace form by the very simple substitution $q = pt$, which leads to

$$J_0(\sigma \sqrt{t}) = \frac{1}{2\pi j} \int_{m-j\infty}^{m+j\infty} \exp\left(pt - \frac{\sigma^2}{4p}\right) \frac{dp}{p} = \mathcal{L}^{-1} \exp\left(-\frac{\sigma^2}{4p}\right) \quad (40)$$

The verification can be had by expanding $J_0(\sigma \sqrt{t})$ into its Taylor series and taking the Laplace transform term by term.

To find the inverse Laplace transform (26) which we need for the current and voltage solutions, we introduce into the inverse Laplace integral

$$\frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{\exp[-u/x \sqrt{(p+\delta)^2 - \sigma^2}]}{\sqrt{(p+\delta)^2 - \sigma^2}} e^{pt} dp \quad (41)$$

new variables similar to what we did in (33). First, we simplify by defining $p + \delta = \rho$

$$\frac{e^{-\delta t}}{2\pi j} \int_{c-j\infty}^{c+j\infty} \exp\left(\rho t - \frac{x}{u} \sqrt{\rho^2 - \sigma^2}\right) \frac{d\rho}{\sqrt{\rho^2 - \sigma^2}}$$

Then we use, in analogy to (34)

$$q = \frac{1}{2} \left(t - \frac{x}{u}\right) (\rho + \sqrt{\rho^2 - \sigma^2}) \quad (42)$$

so that

$$dq = \frac{1}{2} \left(t - \frac{x}{u}\right) \left(1 + \frac{\rho}{\sqrt{\rho^2 - \sigma^2}}\right) d\rho = \frac{q}{\sqrt{\rho^2 - \sigma^2}} d\rho$$

By inversion we obtain

$$\rho = \frac{q}{t - \frac{x}{u}} + \frac{\sigma^2}{4q} \left(t - \frac{x}{u} \right) \quad (43)$$

The exponent of the exponential becomes, if we use (42) for the square root term

$$\rho t - \frac{x}{u} \left(\frac{2q}{t - \frac{x}{u}} - \rho \right) = \rho \left(t + \frac{x}{u} \right) - \frac{2(x/u)}{t - (x/u)} q$$

Inserting ρ from (43) we observe that its first term combined with the second term in this last line gives simply q , so we arrive at

$$q + \frac{\sigma^2}{4q} \left[t^2 - \left(\frac{x}{u} \right)^2 \right]$$

and find the integral form

$$\frac{e^{-\delta t}}{2\pi j} \int_{m-j\infty}^{m+j\infty} \exp \left[q + \frac{\sigma^2}{4q} \left(t^2 - \left(\frac{x}{u} \right)^2 \right) \right] \frac{dq}{q} = e^{-\delta t} I_0 \left[\sigma \sqrt{t^2 - \left(\frac{x}{u} \right)^2} \right] \quad (44)$$

which we identify from (39) quite readily. We have thus demonstrated in accordance with McLachlan,^{D13} p. 217, the validity of (26) and can further verify it with integral relations given in Watson,* see also Doetsch,^{D5} p. 377, or the Campbell-Foster tables,^{E1} pair 860.0. The actual evaluation of integrals of this type is rather involved and often leads to saddle point integration; see references by Courant-Hilbert, Watson, Guillemin, *loc. cit.*

8.4 Response of the Infinite Line to a Sinusoidal Voltage

Suppose we first consider a transmission line with low losses and apply directly at its sending end terminals a sinusoidal voltage

$$v_0(t) = V_m \sin(\omega t + \psi) = \text{Im}(\bar{V} e^{j\omega t}) \quad (45)$$

As we have shown in (10), the voltage wave will progress without distortion, so that the solution is at once

$$\left. \begin{aligned} v(x, t) &= e^{-\delta t} v_0 \left(t - \frac{x}{u} \right) \\ &= e^{-\delta t} V_m \sin \left[\omega \left(t - \frac{x}{u} \right) + \psi \right] \end{aligned} \right\} t > \frac{x}{u} \quad (46)$$

* G. N. Watson, *A Treatise on the Theory of Bessel Functions*, p. 416, Cambridge University Press, England, 1922.

The current wave, however, will suffer distortion because the characteristic impedance is frequency dependent as evident in (6). We can use the complete formal solution (12) for the current, the first term of which is the undistorted wave, whereas the second term requires integration, which gives with the complex notation from (45)

$$\text{Im} \int_{t=x/u}^t \bar{V} e^{j\omega\left(t-\frac{x}{u}\right)} e^{\sigma t} dt = \text{Im} \frac{\bar{V}}{\sigma + j\omega} (e^{\sigma t} e^{j\omega\left(t-\frac{x}{u}\right)} - e^{\frac{\sigma x}{u}})$$

Upon rationalizing and taking the imaginary part only, we get

$$\frac{V_m}{\sigma^2 + \omega^2} e^{\sigma t} \left\{ \sigma \sin \left[\omega \left(t - \frac{x}{u} \right) + \psi \right] - \omega \cos \left[\omega \left(t - \frac{x}{u} \right) + \psi \right] - (\sigma \sin \psi - \omega \cos \psi) e^{-\sigma \left(t - \frac{x}{u} \right)} \right\} \quad (47)$$

so that the complete current response becomes

$$i(x, t) = \frac{\omega^2}{\sigma^2 + \omega^2} \frac{v(x, t)}{R_s} + \frac{\sigma^2}{\sigma^2 + \omega^2} \omega C_s V_m e^{-\sigma t} \cos \left[\omega \left(t - \frac{x}{u} \right) + \psi \right] + \frac{\sigma}{\sigma^2 + \omega^2} \frac{V_m}{R_s} e^{-\frac{\sigma x}{u}} (\sigma \sin \psi - \omega \cos \psi) e^{-\sigma \left(t - \frac{x}{u} \right)} \quad (48)$$

To simplify this form we have combined the first term in (12) with the first term of the integral as we take it from (47), giving the total in-phase steady-state current response; for $\sigma = 0$ it reduces exactly to the distortionless response in section 6.7, which then becomes the total response here also. In the second term of (48), which comes from the second term of the integral as we take it from (47), we have introduced the fictitious series capacitance of the low-loss line from (8), $C_s = (\sigma R_s)^{-1}$, so that it exhibits the exact form of the quadrature current in a parallel admittance branch. The last term in (48) constitutes the local transient caused by the arriving voltage wave because of the apparent "termination" into the characteristic impedance Z_c from (6) or (7) since we are dealing with an infinitely long line.

We recognize this analogy readily if we actually apply the voltage (45) to the series combination of R_s and C_s as shown in Fig. 8.4. For the lumped circuit we have in Laplace transforms combined with complex notation

$$\left(R_s + \frac{1}{pC_s} \right) \bar{I}(p) = \bar{V}(p) = \frac{\bar{V}}{p - j\omega}$$

where $\bar{V} = V_m e^{j\psi}$ with ψ the arbitrary switching angle. The inverse

Laplace transform gives the current in complex notation

$$\bar{i}(t) = \frac{\bar{V}}{R_s} \mathcal{L}^{-1} \frac{p}{p - j\omega} \frac{1}{p + \alpha}$$

with $\alpha = (R_s C_s)^{-1}$, the inverse time constant of the circuit. We

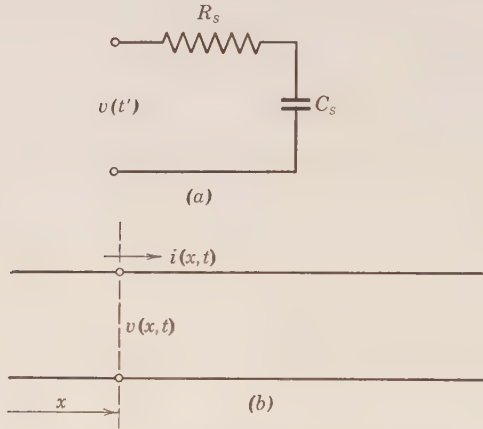


Fig. 8.4. Local equivalent circuit for the low-loss line. (a) Lumped circuit equivalent with $t' = t - (x/u)$; (b) actual transmission-line quantities.

apply the superposition from line 13, Table 1.5, and obtain after rationalization

$$i(t) = \frac{\omega^2}{\alpha^2 + \omega^2} \frac{v(t)}{R_s} + \frac{\alpha^2}{\alpha^2 + \omega^2} \omega C V_m \cos(\omega t + \psi) + \frac{\alpha}{\alpha^2 + \omega^2} \frac{V_m}{R_s} (\alpha \sin \psi - \omega \cos \psi) e^{-\alpha t} \quad (49)$$

This solution is identical in structure with (48) and can be made identical in detail if we replace

$$v(t) \quad \text{by} \quad e^{-\delta t} v_0 \left(t - \frac{x}{u} \right), \text{ from (46)}$$

$$\alpha \quad \text{by} \quad \sigma$$

$$t \quad \text{by} \quad t - \frac{x}{u}$$

which means we interpret t as the local time counted from the arrival of the voltage wave, which is undistorted but attenuated by $\exp[-\delta(x/u)]$. The distortion constant σ is actually the inverse local time constant.

For the general transmission line we cannot proceed so simply. We need to start with the general Laplace transforms of voltage and current given in (24) and (25), respectively, replacing there the Laplace transform of the step voltage V_m/p by that of the sine wave. In complex notation we obtain, e.g., for the current (25)

$$\bar{I}(x, p) = \frac{\bar{V}}{p - j\omega_0} \cdot \frac{p}{V_m} I_i(x, p) = e^{j\psi} \frac{p}{p - j\omega_0} I_i(x, p) \quad (50)$$

where we used ω_0 for the applied angular frequency and where we designated the exact transform (25) by the subscript i to suggest the nature of "indicial" response, i.e., response to standard unit step; see Vol. I, p. 173. If we define $\bar{V} = V_m e^{j\psi}$ as we usually have done, we can simplify to the last form in (50).

We now have several possible lines of attack in (50) to get the inverse Laplace transform, but few are really practical. We could first of all apply the superposition theorem 13 of Table 1.5 since we know the inverse Laplace transform of $I_i(x, p)$, either in the complete form (28) or in the approximation (29). Unfortunately, the resulting integral, even if we take (29) for simplicity

$$\int_{\tau=x/u}^t e^{-\delta\tau} I_0 \left[\sigma \sqrt{\tau^2 - \left(\frac{x}{u} \right)^2} \right] e^{-j\omega_0\tau} d\tau$$

cannot be evaluated in closed form. For very small values of time we could use the convergent series expansion of the Bessel function*

$$I_0(y) = 1 + \frac{(y/2)^2}{(1!)^2} + \frac{(y/2)^4}{(2!)^2} + \frac{(y/2)^6}{(3!)^2} + \cdots \quad (51)$$

However, the integrations are still quite formidable and not obtainable in closed form either. Though for large values of time we have the comparatively simple asymptotic expression (27) for the Bessel function, we can use it only at the large values of t and thus are still faced with the same integral as above for the smaller values of t .

If we write out the inverse Laplace integral for (50) with (25) we have, again in complex notation

$$\bar{i}(x, t) = \frac{u}{2\pi j} V_m e^{j\psi} \int_{c-j\infty}^{c+j\infty} \frac{g + pc}{p - j\omega_0} \frac{e^{-\frac{x}{u}Q}}{Q} e^{pt} dp \quad (52)$$

where Q is defined in (26). This integral has branch points at

$$Q = 0, \quad p = \pm \sigma - \delta = -g/c, -r/l$$

* See references on p. 388.

both located on the negative real axis so that they define the branch cut indicated in Fig. 8.5. In addition, the rational factor contributes a pole at $p = j\omega_0$ which we must accept in the branch where we select the path of integration in order not to lose the contribution of the steady-state response. The closed line integral over path C , the

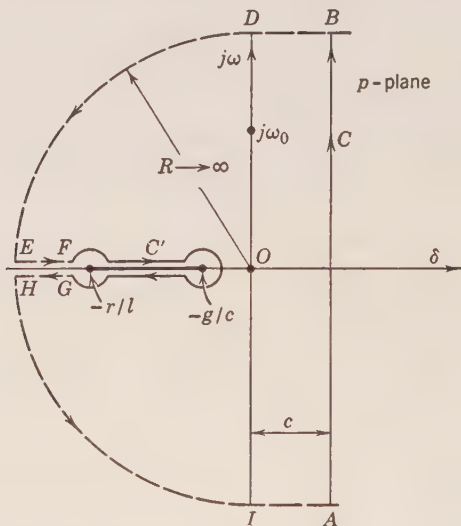


Fig. 8.5. Singularities of the integrand in (52); current response of infinite line to sinusoidal voltage.

dotted lines, and the contour C' around the branch cut thus result in the residue at the pole $p = j\omega_0$, which is

$$uV_me^{j\psi}(g + j\omega_0 c) \frac{e^{-\frac{x}{u}\Gamma}}{\Gamma} e^{j\omega_0 t}$$

with

$$\Gamma^2 = (j\omega_0 + \delta)^2 - \sigma^2 = \left(j\omega_0 + \frac{g}{c}\right) \left(j\omega_0 + \frac{r}{l}\right)$$

We can rationalize this as the steady-state response

$$i_{ss}(t) = \frac{V_m}{\sqrt{l/c}} \left(\frac{(g/c)^2 + \omega_0^2}{(r/l)^2 + \omega_0^2} \right)^{1/2} e^{-\frac{x}{u}\alpha(\omega_0)} \sin[\omega_0 t + \psi - \beta(\omega_0) - \varphi(\omega_0)] \quad (53)$$

where we defined the steady-state propagation function

$$\Gamma = \alpha(\omega_0) + j\beta(\omega_0) \quad (54)$$

in terms of attenuation and phase functions. The phase angle $\varphi(\omega_0)$ arises from the steady-state characteristic impedance

$$Z_c = \left(\frac{r + j\omega_0 l}{g + j\omega_0 c} \right)^{1/2} = \left| \frac{r + j\omega_0 l}{g + j\omega_0 c} \right|^{1/2} e^{j\varphi(\omega_0)} \quad (55)$$

which we could have introduced directly into (53). The amplitude of the steady-state current decreases exponentially along the line.

Returning to the total expression (52), we can now write

$$\phi = i_{ss}(t) = \int_C + \int_{BDEF} + \int_{C'} + \int_{GHIA}$$

so that the inverse Laplace integral reduces to

$$\int_C = i_{ss}(t) - \int_{C'} \quad (56)$$

i.e., the steady-state response from (53) and the reversed contour integral C' around the branch cut. We remember that this is in complete accord with our discussion of the simple inverse Laplace integral (33) which we could replace completely by the reversed contour integral around the branch cut in Fig. 8.3a. But the evaluation of this contour is not simple, in fact, cannot be done in closed form. We could try to expand the exponential in (52) which we could do only around the origin, where $|p|$ is very small; this would lead to an asymptotic expansion in the time variable as discussed in sections 7.3 and 7.4. However, the evaluation of the integrals, though possible, is quite awkward and unrewarding.

Since direct integrations are difficult in this case, we might proceed with Carson* and write the integrand in (52) in the manner indicated in (50), namely

$$\bar{i}(x, t) = e^{j\psi} \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{p}{p - j\omega_0} I_i(x, p) e^{pt} dp \quad (57)$$

If we now expand the first factor under the integral sign into a Taylor series

$$\begin{aligned} \frac{p}{p - j\omega_0} = j \left[\frac{p}{\omega_0} - \left(\frac{p}{\omega_0} \right)^3 + \left(\frac{p}{\omega_0} \right)^5 - \cdots \right] \\ + \left[\left(\frac{p}{\omega_0} \right)^2 - \left(\frac{p}{\omega_0} \right)^4 + \left(\frac{p}{\omega_0} \right)^6 - \cdots \right] \end{aligned} \quad (58)$$

then this series will converge as long as $|p/\omega_0| < 1$. Actually, looking

* J. R. Carson, *Electric Circuit Theory and Operational Calculus*, chapter VII, McGraw-Hill, New York, 1926.

at Fig. 8.5, any such expansion can be valid only for $|p| < g/c$ or $|p/\omega_0| < 1$, whichever is smaller. This means that expansion (58) will lead to an asymptotic development of the time function valid for $t \rightarrow \infty$. We can now readily interpret each of the positive powers in p as differentiations of the inverse Laplace transform of (25) which we have found in (28) or in approximate form in (29). Though we should, in accordance with line 3, Table 1.4, add the function and derivative values at $t = 0^+$, we can dispense with these here since we are only concerned with values of the function for large t . We thus can write for (57) the series

$$\bar{i}_t(x, t) = e^{j\psi} \left[j \left(\frac{1}{\omega_0} \frac{d}{dt} - \frac{1}{\omega_0^3} \frac{d^3}{dt^3} + \frac{1}{\omega_0^5} \frac{d^5}{dt^5} - \dots \right) + \left(\frac{1}{\omega_0^2} \frac{d^2}{dt^2} - \frac{1}{\omega_0^4} \frac{d^4}{dt^4} + \dots \right) \right] i_i(x, t) \quad (59)$$

All the differentiations are to be performed on either (28) or (29), whichever we prefer to use. If we take for simplicity (29), and replace the Bessel function at the same time by its asymptotic form (27), so that

$$i_i(x, t) \approx \frac{V_m}{R_s} \frac{e^{-\gamma t}}{\sqrt{2\pi\sigma t}}, \quad \gamma = \frac{g}{c}, \quad t \text{ large}$$

we can perform the differentiations with less cumbersome detail. After ordering and rationalizing, the final result for the imaginary part of (59) becomes

$$i_t(x, t) = \frac{V_m}{R_s} \frac{e^{-\gamma t}}{\sqrt{2\pi\sigma t}} \left\{ \sin \psi \left[A_2 \left(\frac{\gamma}{\omega_0} \right)^2 - A_4 \left(\frac{\gamma}{\omega_0} \right)^4 + \dots \right] - \cos \psi \left[A_1 \frac{\gamma}{\omega_0} - A_3 \left(\frac{\gamma}{\omega_0} \right)^3 + \dots \right] \right\} \quad (60)$$

with

$$A_1 = \frac{1}{2\gamma t} + 1$$

$$A_2 = \frac{3}{(2\gamma t)^2} + \frac{2}{2\gamma t} + 1$$

$$A_3 = \frac{15}{(2\gamma t)^3} + \frac{9}{(2\gamma t)^2} + \frac{3}{2\gamma t} + 1$$

$$A_4 = \frac{105}{(2\gamma t)^4} + \frac{60}{(2\gamma t)^3} + \frac{18}{(2\gamma t)^2} + \frac{4}{2\gamma t} + 1$$

We clearly recognize the asymptotic character of this development which gives good results for t sufficiently large. A particularly simple form is obtained for the nonleaky line with $g = 0$, where $\delta = \sigma = r/l$ and $\gamma = 0$. In this case

$$i_i(x, t) = \frac{V_m}{R_s} (2\pi\sigma t)^{-1/2}, \quad g = 0, \quad t \text{ large}$$

so that the differentiations lead to the series

$$\begin{aligned} i_t(x, t) = \frac{V_m}{R_s} \frac{1}{\sqrt{2\pi\sigma t}} & \left[\sin \psi \left(\frac{1 \cdot 3}{(2\omega_0 t)^2} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{(2\omega_0 t)^4} \right. \right. \\ & + \cdots (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (4n-1)}{(2\omega_0 t)^{2n}} + \cdots \Big) \\ & - \cos \psi \left(\frac{1}{2\omega_0 t} - \frac{1 \cdot 3 \cdot 5}{(2\omega_0 t)^3} \right. \\ & \left. \left. + \cdots (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (4n+1)}{(2\omega_0 t)^{2n+1}} + \cdots \right) \right] \end{aligned}$$

We see that the a-c transient is definitely smaller than the d-c transient for large values of time. We can also judge that the asymptotic series have their smallest terms when $4n \approx 2\omega_0 t$.

We have carried with (59) and (60) a subscript t to the current without explanation. It should perhaps be obvious now that expansion (58) is about the origin $p = 0$, and therefore cannot include the contribution of the pole $p = j\omega_0$. The total current response for t large must therefore be given by the sum

$$i(x, t) = i_{ss}(x, t) + i_t(x, t) \quad t \text{ large}$$

of (53) and (60), respectively. We also remember that in section 7.4 specific mention was made that an asymptotic expansion cannot portray any oscillatory behavior of a function at large values of time.

There is no room here to discuss in detail the a-c steady-state solution (53) for the current, or

$$v_{ss}(x, t) = V_m e^{-\frac{x}{u}\alpha(\omega_0)} \sin [\omega_0 t + \psi - \beta(\omega_0)] \quad (61)$$

for the voltage. Many of the references in section C, Appendix 3, give rather extensive treatments of the steady-state behavior of transmission lines; see particularly Guillemin,^{C5} for careful discussions of

approximations and Carson and Hoyt* for a detailed analysis of the propagation functions for systems of parallel wires.

8.5 The Finite General Transmission Line

The complete formal solution for the transmission-line voltage and current transforms with arbitrary terminations at both ends as sym-

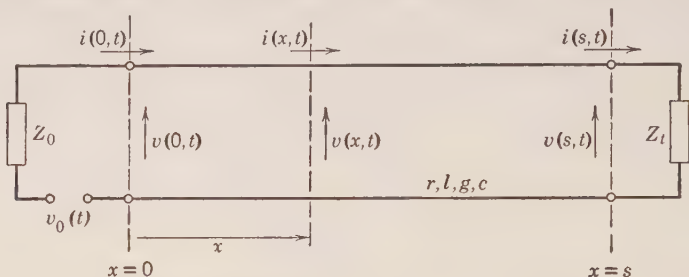


Fig. 8.6. The general transmission line terminated at both ends into arbitrary impedances.

bolized in Fig. 8.6 was deduced in section 6.5 and we repeat here (6.4) and (6.48)

$$V(x, p) = \frac{Z_c}{Z_c + Z_0} \frac{e^{-nx} - \rho_t e^{-n(2s-x)}}{1 - \rho_0 \rho_t e^{-2ns}} \mathcal{L}v_0(t) \quad (62)$$

$$I(x, p) = \frac{1}{Z_c + Z_0} \frac{e^{-nx} + \rho_t e^{-n(2s-x)}}{1 - \rho_0 \rho_t e^{-2ns}} \mathcal{L}v_0(t) \quad (63)$$

where the current reflection coefficients at the two terminations are given by (6.46), namely

$$\rho_0 = \frac{Z_c - Z_0}{Z_c + Z_0}, \quad \rho_t = \frac{Z_c - Z_l}{Z_c + Z_l} \quad (64)$$

The voltage reflection coefficients would be the negative values of these.

For the general transmission line, the propagation function $n(p)$ as well as the characteristic impedance $Z_c(p)$ are irrational functions of p , given specifically in (20) and (21), namely

$$n^2 = (r + pl)(g + pc) = lc[(p + \delta)^2 - \sigma^2] \quad (65)$$

$$Z_c^2 = \frac{r + pl}{g + pc} = \frac{l p + r/l}{c p + g/c} = \frac{n^2}{(g + pc)^2} \quad (66)$$

where the various forms are useful in specific applications.

* J. R. Carson and R. S. Hoyt, "Propagation of Periodic Currents over a System of Parallel Wires," *Bell System Tech. J.*, **6**, 495-545 (1927).

Because of the irrational nature of the characteristic impedance, it is impossible to provide “proper” terminations with lumped-parameter networks, i.e., complete matching of terminal impedances over finite ranges of frequencies. This points at once to the difficulties which are encountered if we try to find rigorous solutions in the general case.

Traveling wave expansion. We can of course expand in a formal way solutions (62) and (63) into the infinite series of exponential functions as we did in (6.51), though we must examine carefully whether we can satisfy

$$|\rho_0\rho_t e^{-2ns}| = \left| \frac{(r + pl) - Z_0Z_t(g + pc) - n(Z_0 + Z_t)}{(r + pl) + Z_0Z_t(g + pc) + n(Z_0 + Z_t)} e^{-2ns} \right| < 1$$

(67)

where the product $\rho_0\rho_t$ was formed with (64) using also (66) in order to show the explicit dependence on n . We can be quite sure that $\text{Re } n > 0$, because we make this the choice in the solution of the infinite line, as we pointed out in (23), and we can choose $\text{Re } n$ large enough to guarantee (67) by choosing our path of integration far enough to the right of the imaginary axis of the p -plane. If we are then entitled to expand binomially, we get for (62) and (63), respectively

$$V(x,p) = \frac{Z_c}{Z_c + Z_0} \mathfrak{L}v_0(t)[e^{-nx} - \rho_t e^{-n(2s-x)} + \rho_0\rho_t e^{-n(2s+x)} \\ - \rho_0\rho_t^2 e^{-n(4s-x)} + \rho_0^2\rho_t^2 e^{-n(4s+x)} - + \cdots]$$

(68)

$$I(x,p) = \frac{1}{Z_c + Z_0} \mathfrak{L}v_0(t)[+ + + + +]$$

(69)

where we indicate in the current transform that the bracket contains the same terms as the voltage bracket, but all with positive signs. This expansion leads obviously again to the concept of traveling waves along the finite line; in fact, each exponential in (68) has the exact form and therefore the identical inverse Laplace transform, line 5 from Table 8.1, replacing x successively by $(2s - x)$, $(2s + x)$, etc. The complications arise, however, from the associated factors if these are functions of p . Unfortunately, it does not help here to take pure resistance as terminations.

The only simple solutions exist if we take

$$Z_0 = 0, \qquad Z_t = 0, \infty$$

so that

$$\rho_0 = 1, \qquad \rho_t = +1, -1$$

If we then also take for the applied voltage the step voltage $V_m 1$, we see that we can write the voltage solution in terms of the solution for the infinite line (31), which we might now designate by $v_\infty(x, t)$. Thus with upper signs for the short-circuited and lower signs for the open-circuited line

$$v(x, t) = v_\infty(x, t) \mp v_\infty(2s - x, t) \pm v_\infty(2s + x, t) - v_\infty(4s - x, t) \\ + v_\infty(4s + x, t) \mp \cdots \quad (70)$$

Each voltage wave is a function of time at any location x of the kind shown in Fig. 8.2 where we had assumed $g = 0$, so that $\delta = \sigma$. Each wave "arrives" at the local starting time which is x/u for the first, $(2s - x)/u$ for the second, $(2s + x)/u$ for the third, etc. Since the waves attenuate continuously during their travel, we can build up with comparatively few terms the complete solution. There will, of course, be considerable distortion caused by the reflections at the ends.

For the current we find the same situation with the same assumptions. Designating solution (28) or (29) with $i_\infty(x, t)$ analogous to the voltage above, we have here with proper attention to the signs, upper for short- and lower for open-circuit of the line

$$i(x, t) = i_\infty(x, t) \pm i_\infty(2s - x, t) \pm i_\infty(2s + x, t) + i_\infty(4s - x, t) \\ + i_\infty(4s + x, t) \pm \cdots \quad (71)$$

and each individual wave is of the type shown in Fig. 8.1.

For any termination other than 0 and ∞ as well as any voltage other than the step voltage, we are often faced with the very unpleasant task of using the general superposition integral from Table 1.5 which can be evaluated only by numerical or graphical means. This requires first the evaluation of the inverse Laplace transform of the reflection coefficients and their products which usually are irrational functions and lead to special integration paths in the complex p -plane such as we discussed in sections 8.3 and 8.4.

Standing wave expansion. We can rewrite (62) and (63) in terms of hyperbolic functions by first multiplying numerator and denominator by e^{ns} and then observing

$$e^{\pm y} = \cosh y \pm \sinh y$$

We obtain

$$V(x, p) = \frac{\sinh n(s - x) + k_t \cosh n(s - x)}{(1 + k_0 k_t) \sinh ns + (k_t + k_0) \cosh ns} \mathcal{L}v_0(t) \quad (72)$$

$$I(x, p) = \frac{\cosh n(s - x) + k_t \sinh n(s - x)}{(1 + k_0 k_t) \sinh ns + (k_t + k_0) \cosh ns} \frac{\mathcal{L}v_0(t)}{Z_c} \quad (73)$$

where we defined

$$k_0 = \frac{Z_0}{Z_c}, \quad k_t = \frac{Z_t}{Z_c} \quad (74)$$

for the significant impedance ratios which give simpler forms in this case than the reflection coefficients. These Laplace transforms invite the application of the residue theorem if we can justify it, i.e., if we can demonstrate that (72) and (73) are actually rational functions as we also showed for the ideal cable in section 7.8. Let us remember from (66) that $Z_c = n/(g + pc)$ and multiply numerator and denominator in (72) by Z_c . With obvious segregation of factors we obtain

$$V(x, p) = \frac{n \sinh n(s - x) + Z_t(g + pc) \cosh n(s - x)}{[r + pl + Z_0 Z_t(g + pc)] \frac{\sinh ns}{n} + (Z_t + Z_0) \cosh ns} \mathcal{L}v_0(t) \quad (75)$$

Since $\cosh y$ is an even function of y , it will only contain powers of y^2 . Similarly, both $y \sinh y$ and $y^{-1} \sinh y$ are even functions of y . This means that the corresponding functions in (75) are functions of n^2 and therefore integral in p and all their coefficients are evidently rational in p if Z_0 and Z_t are lumped-parameter impedances. Thus, in spite of its appearance, the transform $V(x, p)$ in (72) is a meromorphic function in p and we can apply the residue theorem. We can show the same fact about $I(x, p)$ in (73). We can now state in general: Any finite transmission line with lumped-impedance terminations is described by meromorphic Laplace transforms!

Having established the applicability of the expansion or residue theorem, we first need to find the poles of the transform (72) or the roots of its denominator function. By inspection we have as a characteristic equation

$$\tanh ns = - \frac{(Z_t + Z_0)Z_c}{Z_c^2 + Z_0 Z_t} = - \frac{(Z_t + Z_0)ns}{R_l + pL_l + Z_0 Z_t(G_l + pC_l)}$$

where we replaced Z_c by (66) and introduced the total line parameters

$$R_l = rs, \quad L_l = ls, \quad G_l = gs, \quad C_l = cs \quad (76)$$

Of better practical use is the trigonometric form. Taking

$$n = jm, \quad \tanh ns = j \tan ms \quad (77)$$

we get

$$\tan ms = - \frac{Z_t + Z_0}{R_l + pL_l + Z_0 Z_t(G_l + pC_l)} ms \quad (78)$$

The only possible way to find the roots of this transcendental equation is by graphical or numerical methods. If we assume at first that ms is real, then we can plot the left-hand side as a regular tangent graph; the intersections of the infinity of branches of $\tan ms$ with the higher order curve defined by the right-hand side of (78) as a function of ms will then give the root values of ms and by tracing back through (77) and (65) we get the root values of p , namely

$$\left. \begin{matrix} p_{\alpha}' \\ p_{\alpha}'' \end{matrix} \right\} = -\delta \pm \sqrt{\sigma^2 - (m_{\alpha}u)^2} = -\delta \pm j \sqrt{(m_{\alpha}u)^2 - \sigma^2} \quad (79)$$

For every root value m_{α} we find two negative real or conjugate complex values of p_{α} , depending on whether $m_{\alpha}u \leq \sigma$.

We expect an infinity of root values because $\tan ms$ belongs to the class of *meromorphic* functions which we discussed in connection with (6.84) as well as at the end of section 3.2.

It is, of course, necessary to check that none of the roots of the denominator also occur in the numerator function so as to remove the respective pole of $V(x, p)$ or $I(x, p)$. It is also important to check the poles of the voltage transform $\mathcal{L}v_0(t)$ or the corresponding general source function. Should it possess a pole identical with one of the numerator, then we again have cancellation; should it, on the other hand, possess a pole in common with the denominator function, then we need to consider the multiplicity of the pole in the evaluation of the residue.

Example 1, $Z_0 = Z_t = 0$. As a simple illustration we might take the general line with $Z_0 = Z_t = 0$, short-circuited at both ends, with a step voltage $V_m 1$ applied at $t = 0$. The transform (72) is in this case

$$V(x, p) = \frac{\sinh n(s-x)}{\sinh ns} \frac{V_m}{p} \quad (80)$$

very similar to the lossless line in (6.84). In fact, we can take the roots of the denominator from (6.85) with the appropriate adjustment

$$\sinh ns = 0, \quad n_{\alpha} = j \frac{\alpha\pi}{s}, \quad \alpha = 0, \pm 1, \pm 2, \dots \quad (81)$$

We observe that $n_0 = 0$ is also a root of the numerator, so it does not represent a pole of (80). The corresponding root values of p are given by (79), if we use (77)

$$m_{\alpha} = \frac{\alpha\pi}{s}, \quad p_{\alpha}', p_{\alpha}'' = -\delta \pm j \sqrt{\left(\frac{\alpha\pi u}{s}\right)^2 - \sigma^2} = -\delta \pm j \Omega_{\alpha} \quad (82)$$

Obviously, $p = 0$ is not a root of the denominator, nor of the numerator, of the first fraction and must be retained. On the other hand, the \pm signs of α in (81) do not produce different values of p_α in (82) so that only positive values need be considered.

In accordance with (1.57) or line 12 in Table 1.4, we can construct the total response. We have for the denominator of (80)

$$\frac{d}{dp} \sinh ns = s \cosh ns \frac{dn}{dp} = \pm \frac{s}{nu^2} \sqrt{(nu)^2 + \sigma^2} \cosh ns \quad (83)$$

where we deduced the derivative of n from (65), namely

$$2n \frac{dn}{dp} = \frac{1}{u^2} 2(p + \delta), \quad \frac{dn}{dp} = \frac{p + \delta}{nu^2} = \pm \frac{1}{nu^2} \sqrt{(nu)^2 + \sigma^2} \quad (84)$$

The two signs of $(p + \delta)$ correspond to the two root values from (65). The root $p = 0$ entrains from (65)

$$n_0 = \sqrt{lc} \sqrt{\delta^2 - \sigma^2} = \sqrt{rg}$$

The roots (81) give for the numerator in (80)

$$\sinh n_\alpha(s - x) = -(-1)^\alpha j \sin \frac{\alpha\pi x}{s}$$

and for the denominator part (83)

$$\pm \frac{s^2}{(n_\alpha s)u^2} j \Omega_\alpha \cosh n_\alpha s = \pm j \frac{\Omega_\alpha s^2}{\alpha\pi u^2} (-1)^\alpha$$

The total response becomes

$$v(x, t) = V_m \frac{\sinh \sqrt{rg} (s - x)}{\sinh \sqrt{rg} s} + V_m \sum_{\alpha=1}^{\infty} \frac{-(-1)^\alpha j \sin [(\alpha\pi x)/s] \alpha\pi u^2}{(-\delta \pm j\Omega_\alpha) [\pm j\Omega_\alpha s^2 (-1)^\alpha]} e^{(-\delta \pm j\Omega_\alpha)t}$$

The upper and lower signs in the sum belong to $p_{\alpha'}$ and $p_{\alpha''}$, respectively, so that for each α we have to form the subsum of the terms with double sign. Keeping out the common factors, we combine

$$+ \frac{e^{j\Omega_\alpha t}}{-\delta + j\Omega_\alpha} - \frac{e^{-j\Omega_\alpha t}}{-\delta - j\Omega_\alpha} = \frac{2j \sin (\Omega_\alpha t + \varphi_\alpha)}{\sqrt{\delta^2 + \Omega_\alpha^2}}$$

where

$$\varphi_\alpha = \tan^{-1} \frac{\Omega_\alpha}{\delta}$$

The final form is thus

$$v(x,t) = V_m \frac{\sinh \sqrt{rg} (s-x)}{\sinh \sqrt{rg} s} + V_m \frac{2\pi u^2}{s^2} e^{-\delta t} \sum_{\alpha=1}^{\infty} \frac{\alpha \sin [(\alpha\pi x)/s]}{\Omega_{\alpha}(\delta^2 + \Omega_{\alpha}^2)^{1/2}} \sin (\Omega_{\alpha} t + \varphi_{\alpha}) \quad (85)$$

The first term is the steady-state distribution of the voltage along the line; the second term is the superimposed transient response composed of an infinite number of harmonic oscillations because we have assumed that all $m_{\alpha}u > \sigma$, so that we have comparatively low losses. Should the lowest value or $m_{1u} < \sigma$, then we have two real values of p_1' and p_1'' leading to simple exponential transients; these would have to be taken outside the sum in (85) as extra terms and the sum then started with $\alpha = 2$.

Example 2, $Z_0 = 0$, $Z_t \neq 0$. Let us now take a finite line as in Fig. 8.6, apply again a step voltage $V_m 1$, and terminate at the far end into $Z_t = R + pL$, keeping $Z_0 = 0$ at the sending end. With $k_0 = 0$, we can actually write (72) and (73) somewhat simpler, if we expand the numerator by the conventional hyperbolic function relations, namely

$$V(x,p) = \frac{1}{A'} (A' \cosh nx - B' \sinh nx) \mathfrak{L}v_0(t)$$

$$I(x,p) = \frac{1}{A'Z_c} (B' \cosh nx - A' \sinh nx) \mathfrak{L}v_0(t)$$

where

$$A' = \sinh ns + k_t \cosh ns$$

$$B' = \cosh ns + k_t \sinh ns$$

We see that the denominator function is now simpler and the poles of the Laplace transforms can be found from

$$A' = 0, \quad \tanh ns = -k_t = -\frac{Z_t}{Z_c}$$

or if we use for Z_c the last form in (66) and convert by (76) and (77)

$$\tan ms = -\frac{Z_t}{R_l + pL_l} ms = -\rho ms \quad (86)$$

This is the general characteristic equation for $Z_0 = 0$, which we could, of course, also get from (78). With it we can write the transforms in

terms of the parameter m throughout, namely

$$V(x, p) = \frac{1}{A} (A \cos mx - B \sin mx) \mathcal{L}v_0(t) \quad (87)$$

$$I(x, p) = -\frac{G_l + pC_l}{Ams} (B \cos mx + A \sin mx) \mathcal{L}v_0(t) \quad (88)$$

where

$$\begin{aligned} A &= \sin ms + \rho ms \cos ms \\ B &= \cos ms - \rho ms \sin ms \\ \rho &= \frac{Z_l}{R_l + pL_l} \end{aligned} \quad (89)$$

Returning to our specific example $Z_l = R + pL$, we might choose the parameters so as to make ρ a constant, i.e.

$$R = kR_l, \quad L = kL_l, \quad \rho = k = \frac{1}{\pi}$$

and specifically to make it equal to $1/\pi$. The determination of the root values of (86) is shown in the graph of Fig. 8.7 where the left-hand

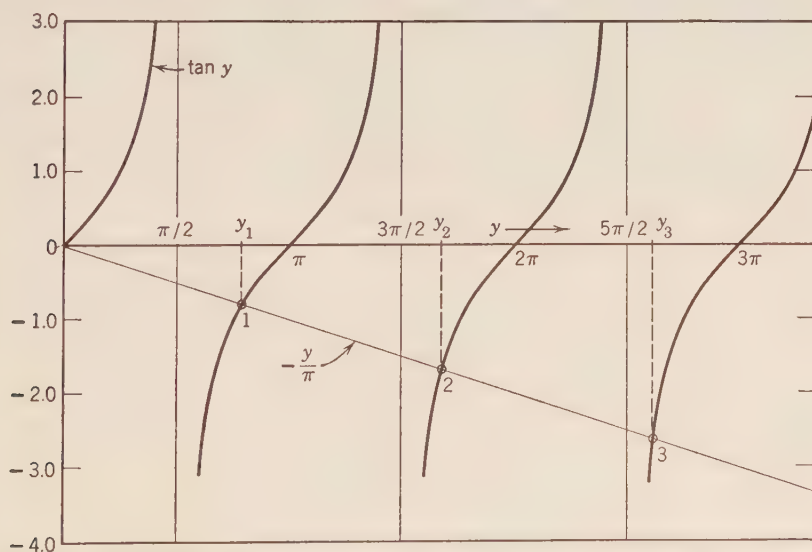


Fig. 8.7. Graphical determination of the roots of the transcendental equation $\tan y = -y/\pi$.

side of (86) is represented by the uniformly recurrent branches $\tan y$, whereas the right-hand side is represented by the straight line $-y/\pi$.

The intersections lead to the root values of which the first three are

$$y = 2.474 = m_1 s, \quad p_1', p_1'' = -\delta \pm \sqrt{\sigma^2 - (m_1 u)^2} = -61, -1139$$

$$y = 5.252 = m_2 s, \quad p_2', p_2'' = -\delta \pm \sqrt{\sigma^2 - (m_2 u)^2} = -213, -987$$

$$y = 8.220 = m_3 s, \quad p_3', p_3'' = -\delta \pm j \sqrt{(m_3 u)^2 - \sigma^2} = -600 \pm j354$$

The numerical values used for the p roots are approximately the same as the last column in Table 6.1

$$\delta = 600 \text{ sec}^{-1}, \quad \sigma = 575 \text{ sec}^{-1}, \quad u = 20,000 \text{ miles/sec}$$

and we chose $s = 250$ miles. We see that the first two values m_1, m_2 lead to four negative real root values p_α , whereas all higher values m_α , $\alpha > 2$ lead to conjugate complex root pairs p_α . From Fig. 8.7 we also infer that as $y > 100$,

$$y_\alpha = m_\alpha s \approx (2\alpha - 1) \frac{\pi}{2}, \quad p_\alpha', p_\alpha'' \approx -\delta \pm j \sqrt{\left(\frac{(2\alpha - 1)\pi u}{2s}\right)^2 - \sigma^2}$$

We can now apply the expansion theorem, line 12 in Table 1.4, directly to (87) if we take $\mathcal{L}v_0(t) = V_m/p$. For the root $p = 0$ of the voltage transform, which does not occur elsewhere in numerator or denominator of the transform, we have from (65) as in the first example

$$p = 0, \quad n_0 = \sqrt{lc} \sqrt{\delta^2 - \sigma^2} = \sqrt{rg}$$

so that the steady-state term becomes, going back to hyperbolic functions

$$v_{ss}(x, t) = V_m \left(\cosh \sqrt{rg} x - \frac{\cosh \sqrt{rg} s + \sqrt{rg} (s/\pi) \sinh \sqrt{rg} s}{\sinh \sqrt{rg} s + \sqrt{rg} (s/\pi) \cosh \sqrt{rg} s} \sinh \sqrt{rg} x \right) \quad (90)$$

For the transient terms we need to evaluate the sum over all roots

$$V_m \sum \left[\frac{-B \sin mx}{p[(dA/dm)(dm/dp)]} \right]_{p_\alpha} e^{p_\alpha t}$$

The numerator and dA/dm only depend on m , so we can directly introduce the numerical root values m_α . The other factors depend on the root values of p_α so that we have two values for each m_α as in the first example which we can compute directly from (79), using the first form for the first two roots and the second form for all higher roots. We

then get for the transient

$$v_t(x, t) = V_m \sum_{\alpha=1}^{\infty} - \left(\frac{B \sin mx}{dA/dm} \right)_{m=m_\alpha} \left[\left(\frac{1}{p} \frac{dp}{dm} e^{pt} \right)_{p=p_\alpha'} + \left(\frac{1}{p} \frac{dp}{dm} e^{pt} \right)_{p=p_\alpha''} \right] \quad (91)$$

Specifically, we need to replace B from (89), and get also from (89)

$$\begin{aligned} \frac{dA}{dm} &= +s \cos ms - \rho ms^2 \sin ms + \rho s \cos ms \\ &= sB + \rho s \cos ms \end{aligned}$$

and from (84), since $n^2 = -m^2$

$$\frac{dm}{dp} = \frac{p + \delta}{-mu^2} = \mp \frac{1}{mu^2} \sqrt{\sigma^2 - (mu)^2} = \mp \frac{j}{mu^2} \sqrt{(mu)^2 - \sigma^2}$$

where we utilized (65) and (79).

For the real root value pairs, the bracket in (91) gives

$$\left(-\frac{1}{p'} e^{p't} + \frac{1}{p''} e^{p''t} \right) \frac{mu^2}{\sqrt{\sigma^2 - (mu)^2}} \quad m_\alpha, p_\alpha; \alpha = 1, 2$$

and for the conjugate complex pairs, the bracket in (91) gives

$$\left(-\frac{e^{j\Omega_\alpha t}}{-\delta + j\Omega_\alpha} + \frac{e^{-j\Omega_\alpha t}}{-\delta - j\Omega_\alpha} \right) e^{-\delta t} \frac{m_\alpha u^2}{j \sqrt{(m_\alpha u)^2 - \sigma^2}} \quad \alpha > 2$$

This has also been evaluated in example 1.

8.6 The Fourpole Aspects of the Transmission Line

The two-wire transmission line can very naturally be considered as a fourpole and, indeed, as a cascade arrangement of pieces of transmission line, each bounded by parallel planes which are normal to the extension of the transmission line. Actually, when we introduced the transmission-line concept in section 6.1, we started from the previously established fourpole notation for lumped-parameter networks, see also Fig. 6.1, and applied it to extremely small lengths Δx of the line.

To deduce the general fourpole parameters for the finite line of length s as indicated in Fig. 8.8a, let us start from the general solution of the differential equation (6.26) for the Laplace transform of the voltage

$$V(x, p) = A e^{nx} + B e^{-nx} \quad (92)$$

and for current in the form (6.29)

$$I(x, p) = \frac{1}{Z_c} (-Ae^{nx} + Be^{-nx}) \quad (93)$$

where Z_c and n are the general characteristic impedance and propagation function as defined in (65) and (66). We will determine the integration constants A and B now from the boundary conditions set

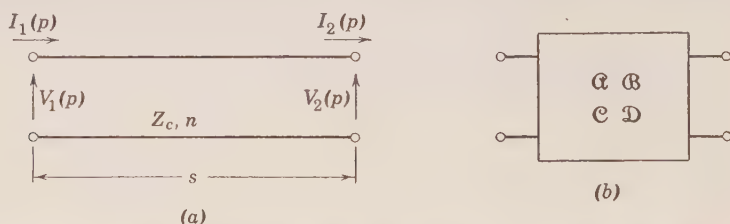


Fig. 8.8. The general transmission line as a fourpole. (a) The finite line; (b) the equivalent fourpole.

down in Fig. 8.8a, namely in terms of input and output voltage-current pairs. The conditions at $x = 0$ are

$$\begin{aligned} V(0, p) &= A + B = V_1(p) \\ I(0, p) &= \frac{1}{Z_c} (-A + B) = I_1(p) \end{aligned} \quad (94)$$

We solve now for A and B in terms of $V_1(p)$ and $I_1(p)$, giving

$$A = \frac{1}{2}(V_1 - Z_c I_1), \quad B = \frac{1}{2}(V_1 + Z_c I_1)$$

If we introduce these results into (92) and (93) and at once combine the appropriate exponentials into hyperbolic functions, we obtain

$$\begin{aligned} V(x, p) &= V_1 \cosh nx - Z_c I_1 \sinh nx \\ I(x, p) &= -\frac{V_1}{Z_c} \sinh nx + I_1 \cosh nx \end{aligned} \quad (95)$$

Letting here $x = s$, and observing

$$V(s, p) = V_2(p), \quad I(s, p) = I_2(p)$$

we arrive at a typical fourpole relation

$$\begin{aligned} V_2(p) &= V_1(p) \cosh ns - Z_c I_1(p) \sinh ns \\ I_2(p) &= -\frac{V_1(p)}{Z_c} \sinh ns + I_1(p) \cosh ns \end{aligned} \quad (96)$$

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Surely, the uniform transmission line is a symmetrical fourpole, so that in analogy to (2.20a) we can identify the general parameters as follows

$$\begin{aligned}\mathfrak{A} &= \cosh ns & \mathfrak{B} &= Z_c \sinh ns \\ \mathfrak{C} &= \frac{1}{Z_c} \sinh ns & \mathfrak{D} &= \cosh ns\end{aligned}\quad (97)$$

As a passive network, the determinant must be unity as in (19), and, indeed, we find

$$\mathfrak{A}\mathfrak{D} - \mathfrak{B}\mathfrak{C} = \cosh^2 ns - \sinh^2 ns = 1$$

If we invert the system (96), we therefore get

$$\begin{bmatrix} V_1(p) \\ I_1(p) \end{bmatrix} = \begin{bmatrix} \cosh ns & Z_c \sinh ns \\ \frac{1}{Z_c} \sinh ns & \cosh ns \end{bmatrix} \times \begin{bmatrix} V_2(p) \\ I_2(p) \end{bmatrix} \quad (98)$$

the standard fourpole relation used early in power system analysis, as we pointed out in section 2.1, particularly on p. 53.

The fourpole notation is particularly useful in the interconnection of transmission lines and lumped-parameter networks in an iterative manner such as in periodic loading of transmission lines, in communication lines with periodically spaced repeaters, and in wave guides with periodic discontinuities.

For the fourpole line we had defined a propagation function Γ by (3.20), which we identify for the transmission line from (97) by

$$\cosh \Gamma = \mathfrak{A} = \cosh ns, \quad \Gamma = ns$$

whereas the characteristic impedance from (3.21) has the identical meaning in the transmission line.

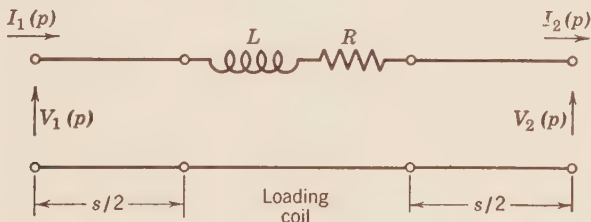


Fig. 8.9. A general transmission-line section with loading coil.

Let us consider the insertion of a loading coil with inductance L and resistance R into the center of a transmission-line length s as shown in Fig. 8.9. Using (97) for the equivalent fourpoles of the two half

lengths of line, i.e., replacing s by $s/2$, we need to form the matrix product for the cascade of the three fourpoles

$$\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{s/2} \times \begin{bmatrix} 1 & R + pL \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}_{s/2}$$

which gives upon multiplication and taking into consideration the relations for hyperbolic functions

$$\cosh^2 n \frac{s}{2} + \sinh^2 n \frac{s}{2} = \cosh ns, \quad 2 \cosh n \frac{s}{2} \sinh n \frac{s}{2} = \sinh ns$$

the over-all matrix of standard parameters

$$\begin{bmatrix} \cosh ns + \frac{R + pL}{2Z_c} \sinh ns & Z_c \sinh ns + (R + pL) \cosh^2 n \frac{s}{2} \\ \frac{1}{Z_c} \sinh ns + \frac{R + pL}{Z_c^2} \sinh^2 n \frac{s}{2} & \cosh ns + \frac{R + pL}{2Z_c} \sinh ns \end{bmatrix}$$

Again we have symmetry as expected. We can now deduce the over-all propagation function Γ from (3.20) on the basis of the fourpole equivalence

$$\cosh \Gamma = \cosh ns + \frac{R + pL}{2Z_c} \sinh ns \quad (99)$$

which indicates the direct effect of the loading coil upon the signal propagation.

The spacing of the loading coils must be at reasonably short intervals so that we can replace in good approximation

$$\sinh ns \approx ns$$

Taking now also the last form of (21) for the characteristic impedance, we can combine in the last term of (99)

$$\frac{1}{2} (R + pL) \frac{ns}{n} (g + pc) = \frac{1}{2} (R + pL) (G_l + pC_l)$$

also using (76) for the total line parameters. To further simplify we need to introduce for the cosh functions the half arguments

$$\cosh \Gamma = 1 + 2 \sinh^2 \frac{\Gamma}{2}$$

and similarly for $\cosh ns$, except that we also make use of the approxi-

mation for $\sinh n(s/2)$

$$\begin{aligned}\cosh ns &= 1 + 2 \sinh^2 n \frac{s}{2} \approx 1 + 2 \left(\frac{ns}{2} \right)^2 \\ &= 1 + \frac{1}{2}(R_l + pL_l)(G_l + pC_l)\end{aligned}$$

The final result is thus

$$4 \sinh^2 \frac{\Gamma}{2} = (G_l + pC_l)(R_l + pL_l + R + pL) \quad (100)$$

We see that in good approximation, for reasonably close spacing of the loading coils, their effect is to add the coil resistance and inductance directly to the total pertinent line parameters. Or in other words, we can assume that the line parameters are modified to

$$r' = r + \frac{R}{s}, \quad l' = l + \frac{L}{s}$$

The loaded line can therefore be treated approximately as a smooth transmission line with

$$\delta = \frac{1}{2} \left(\frac{r'}{l'} + \frac{g}{c} \right), \quad \sigma = \frac{1}{2} \left(\frac{r'}{l'} - \frac{g}{c} \right)$$

We also observe the importance to achieve

$$\frac{l'}{l} \gg \frac{r'}{r} \quad \text{or} \quad \frac{L}{ls} \gg \frac{R}{rs}$$

in order to get close to distortionless conditions as pointed out in section 6.3.

PROBLEMS

8.1 An infinite line with low losses as treated in section 8.1 and defined in (5) and (6) has applied at the sending end a sawtooth pulse of voltage. Evaluate the current at distance x from the sending end.

8.2 The same line as in problem 1 is terminated into a series resistance R at the sending end. A unit step voltage is applied in series with this resistance R . Find the voltage and current build-up at distance x from the sending end.

8.3 A finite line with low losses (see section 8.1) is connected directly to an infinite distortionless cable with the same surge resistance R_s . A step voltage $V_m 1$ is applied to the low loss line. (a) Find the voltage and current solution on the low loss line in terms of traveling waves. (b) Find the first two outgoing voltage and current waves on the distortionless line.

8.4 In problem 8.3, find the current and voltage distribution on the low loss line by the method of standing waves.

8.5 A low loss line of length $2s$ has inserted at the center $x = s$ a series inductance L . A square-wave voltage pulse is applied at the sending end of the line. (a) Find the first voltage and current pulse entering the second half of the line. (b) Find the voltage distribution on the first half line up to the third reflection. (c) Find the steady-state current distribution on the open-circuited line.

8.6 For the far end short-circuited in problem 8.5, find the distribution of voltage and current over the total line by means of the standing wave method.

8.7 A low loss line of length s with characteristics δ_1 and σ_1 is connected to a second low loss line with characteristics δ_2 and σ_2 . At the junction a capacitance C is placed in shunt across the lines. (a) Find the complete Laplace transform of the voltage wave entering line 2 if a step voltage $V_m 1$ is applied at the sending end terminals of the first line. (b) Find the first two traveling voltage waves entering the second line. (c) Find the first two current waves entering the second line.

8.8 An infinite low loss line has a sinusoidal voltage $V_m \cos \omega t$ applied at its sending end terminals. (a) Find the current wave along the line. (b) Find the steady-state voltage and current distribution along the line.

8.9 A finite low loss line is terminated into a resistance $R = R_s$. A sinusoidal voltage $V_m \sin \omega t$ is applied directly at its sending end terminals. (a) Find the voltage and current traveling waves. (b) Find the steady-state voltage and current distribution along the line.

8.10 A finite low loss line is connected to a general transmission line with all four parameters. We assume that $R_s = \sqrt{l/c}$ and σ are the same in both cases. (a) Find the entering voltage and current in the general line, if a step voltage is applied at the sending end terminals of the low loss line. (b) Find the steady-state distribution of current and voltage along the general transmission line.

8.11 An exponential voltage $V e^{-\alpha t} 1$ is suddenly applied to the sending end terminals of an infinite general transmission line. Find the entering current.

8.12 A general transmission line with $g = 0$ is terminated at the far end into a capacitance C . A step voltage is applied at the sending end terminals. (a) Find the voltage distribution along the line by the standing wave method. (b) Find the current distribution.

8.13 In problem 8.12 use the traveling wave concept and (a) find the first current wave arriving at the end of the line, (b) find the initial charge build-up in the capacitance for the time interval $0 < t < 4s/u$.

8.14 The general line in Fig. 8.6 has $Z_0 = 0$ and $Z_t = \infty$. (a) Evaluate the voltage distribution along the line by the standing wave method. (b) Evaluate the first voltage wave arriving at the far end and its first reflection by the traveling wave method. (c) Compare the two developments for $0 < t < 2s/u$.

8.15 The general line in Fig. 8.6 with $Z_0 = Z_t = 0$ has applied to it a sinusoidal voltage $V_m \sin \omega t$ at the sending end. Evaluate the voltage and current distributions along the line by the standing wave method.

8.16 The general line in Fig. 8.6 is connected at the far end to a distortionless line. (a) Find the first voltage wave traveling into the distortionless line if a square-wave voltage pulse is applied at the sending end terminals ($Z_0 = 0$) of the general line. (b) Find the voltage distribution on the general line if the surge resistances of both lines are the same (however, $Z_c \neq R_s$ for the general line).

APPENDIX 1

NOTATION AND SYMBOLS

1. List of Symbols in Alphabetical Order of the Quantities

<i>Symbol</i>	<i>Designation</i>	<i>Remarks</i>
$Y(p), y(p)$	admittance function, parametric (generalized)	
$Y(j\omega), y(j\omega)$	admittance phasor	$Y = G + jB$
$\alpha(\omega)$	attenuation function	
ω, Ω	angular frequency	
C, c	capacitance	
p	complex variable, "complex frequency"	$p = \delta + j\omega$
G, g	conductance	
k	coupling coefficient	
$i(t)$	current, real time function	$i(t) = \text{Im } \bar{i}(t)$
I	current phasor	
$I(p)$	current, Laplace transform	$I(p) = \mathcal{L}i(t)$
$\bar{I}(\omega)$	current, Fourier transform	$\bar{I}(\omega) = \mathcal{F}i(t)$
Ω_c	cutoff frequency	
δ, α	damping coefficient	
σ	distortion constant (lines)	
V_{imp}	"electromotive force," source voltage	
f, F	frequency	
ω, Ω	frequency, angular	
j	imaginary unit	$j = \sqrt{-1}$
$Z(p), z(p)$	impedance function, parametric (generalized)	
$Z(j\omega), z(j\omega)$	impedance phasor	$Z = R + jX$
$A(t)$	indicial admittance	
L, l	inductance	
M, L_m	inductance, mutual	
σ	leakage coefficient	$\sigma = 1 - k^2$
s	length of line	
Λ	linkage, magnetic	
Φ_m	magnetic flux	
N, N_M, N_P	number of turns, of meshes, of node pairs	
T	period	
φ, ψ	phase angle	
$\beta(\omega)$	phase function	
$\Gamma(p), \Gamma(j\omega)$	propagation function (filter)	$\Gamma(j\omega) = \alpha(\omega) + j\beta(\omega)$

<i>Symbol</i>	<i>Designation</i>	<i>Remarks</i>
$n(p), n(j\omega)$	propagation function (lines)	
R, r	resistance	
I_{imp}	source current	
V_{imp}	source voltage, "electromotive force"	
t	time	
T	time constant	
θ	time, relative	$\theta = t/T$
τ	time shift	
$H(p), H(j\omega)$	transfer function	$H(j\omega) = h(\omega)e^{-j\Phi(\omega)}$
Ω_0	undamped free angular frequency	$\Omega_0 = 2\pi F_0$
u	velocity of propagation	
$v(t)$	voltage, real time function	$v(t) = \text{Im } \bar{v}(t)$
V	voltage phasor	
$V(p)$	voltage, Laplace transform	$V(p) = \mathcal{L}v(t)$
$\bar{V}(\omega)$	voltage, Fourier transform	$\bar{V}(\omega) = \mathcal{F}v(t)$

2. Alphabetic List of Symbols

$A(t)$	indicial admittance	
C, c	capacitance	
f, F	frequency	
G, g	conductance	
$H(p), H(j\omega)$	transfer function	$H(j\omega) = h(\omega)e^{-j\Phi(\omega)}$
$I_{\text{imp}}, i(t), I$	current impressed, instantaneous, phasor	
$I(p)$	current transform, Laplace	$I(p) = \mathcal{L}i(t)$
$\bar{I}(\omega)$	current transform, Fourier	$\bar{I}(\omega) = \mathcal{F}i(t)$
$J(t)$	indicial impedance	
j	imaginary unit	$j = \sqrt{-1}$
k	coupling coefficient	
L, l	inductance	
M	mutual inductance	
N, N_M, N_P	number of turns, of meshes, of node pairs	
$n(p), n(j\omega)$	propagation function (lines)	
p	"complex frequency," complex variable	$p = \delta + j\omega$
R, r	resistance	
s	length of line	
t	time	
T	time constant; period	
u	velocity of propagation	
$V_{\text{imp}}, v(t), V$	voltage impressed, instantaneous, phasor	
$V(p)$	voltage transform, Laplace	$V(p) = \mathcal{L}v(t)$
$\bar{V}(\omega)$	voltage transform, Fourier	$\bar{V}(\omega) = \mathcal{F}v(t)$
$Y(p), Y(j\omega)$	admittance function	$Y(j\omega) = G + jB$
$Z(p), Z(j\omega)$	impedance function	$Z(j\omega) = R + jX$
$\alpha(\omega)$	attenuation function	
$\beta(\omega)$	phase function	
$\Gamma(p), \Gamma(j\omega)$	propagation function (filters)	$\Gamma(j\omega) = \alpha(\omega) + j\beta(\omega)$
δ	damping coefficient	
θ	relative time	$\theta = t/T$
Λ	magnetic linkage	

<i>Symbol</i>	<i>Designation</i>	<i>Remarks</i>
σ	leakage coefficient	$\sigma = 1 - k^2$
σ	distortion constant (lines)	
τ	time shift, pulse duration	
φ	phase angle	
Φ_m	magnetic flux	
ψ	phase angle	
ω, Ω	angular frequency	
Ω_0	undamped free angular frequency	
Ω_c	cutoff frequency	

3. General Principles

The use of complex notation in transient analysis requires careful differentiation between complex values arising from the signal notation and complex values introduced by the method of analysis. Thus we can introduce the real or the complex signal current

$$i(t) = I_m \sin (\omega t - \varphi) = \text{Im} (Ie^{j\omega t})$$

$$\bar{i}(t) = Ie^{j\omega t}, \quad I = I_me^{-j\varphi}$$

though the physically observable value will always be the real time function $i(t)$. If we take the Laplace transform of the real time function, we obtain a function in the complex variable $p = \delta + j\omega$ with *real* coefficients; if we take it of the complex time function $\bar{i}(t)$, we get in general complex coefficients. If we use complex signal notation, we must define as answer to the problem the imaginary part of the complex result. If we use real signal notation, the answer must come out in real terms, no matter which method is used.

The complex function of a real variable t is designated by a bar above the symbol, e.g.,

$$\bar{i}(t) = \text{Re } \bar{i}(t) + j \underbrace{\text{Im } \bar{i}(t)}_{i(t)}$$

where $i(t)$ is the actual physical solution which we normally use in transient problems. This principle applies also to the Fourier transform

$$\bar{I}(\omega) = \mathcal{F}i(t) = \frac{1}{\pi} [\Phi_i(\omega) - j\Psi_i(\omega)]$$

where Φ_i and Ψ_i are the real cosine and sine spectrum functions of the current $i(t)$.

The notation used consistently in this book is tabulated below. To avoid using many different symbols for related concepts, the indication

of the functional variable is employed to differentiate between these concepts.

<i>General Time Functions</i>		<i>Symbols</i>		<i>Interrelations</i>
Instantaneous values, real	$i(t)$	$v(t)$	}	$i(t) = \text{Im } \bar{v}(t)$
Complex notation for instantaneous values	$\bar{v}(t)$	$\bar{v}(t)$		
Fourier transform of $i(t)$	$\bar{I}(\omega)$	$\bar{V}(\omega)$		
Laplace transform of $i(t)$	$I(p)$	$V(p)$		
Laplace transform of $\bar{v}(t)$	$\bar{I}(p)$	$\bar{V}(p)$		$\bar{I}(p) = \mathfrak{L}\bar{v}(t)$
<i>Sinusoidal (Harmonic) Time Functions</i>				
Real notation	$I_m \sin (\omega t - \varphi)$	$V_m \sin (\omega t - \varphi)$	}	$i(t) = \text{Im } (Ie^{j\omega t})$
Complex harmonic time functions	$Ie^{j\omega t}$	$Ve^{j\omega t}$		
Phasor (complex amplitude)	I	V		$I = I_me^{-j\varphi}$
Maximum value (real amplitude)	I_m	V_m		$I_m = I $
Root-mean-square value (rms)	I	V		$I = I_m/\sqrt{2}$

APPENDIX 2

GLOSSARY OF SOME MATHEMATICAL TERMS

1. Functions

If to every value of x between any two real numbers a and b , where $b > a$, there corresponds a number $y = f(x)$, then y is a function of x . The numbers a and b constitute the *interval* or domain of x , over which the function is defined. If a and b are excluded, the interval is *open*; if one or the other or both are included, the interval is *closed* at the respective ends. The open interval is designated by $a < x < b$, the closed interval by $a \leq x \leq b$.

2. Continuity

A function $f(x)$ is continuous at a point x_0 , if one can choose a value ϵ ever so close to x_0 and find $f(x)$ ever so close to $f(x_0)$; in mathematical terms, if

$$|f(x) - f(x_0)| < \epsilon \text{ (very small) for any } |x - x_0| \leq \eta \text{ (very small)}$$

If the function is continuous at every point of an interval a, b , then the function is called *continuous* in that interval. If this is true only for subintervals, but does not hold at a finite number of points in a, b , the function is *sectionally continuous*, i.e., continuous in each of the subintervals.

3. Discontinuity

At any point x_0 , where the function is not continuous, it has a *point of discontinuity*. A *finite discontinuity* exists if the function approaches distinct and different values as we approach x_0 from right and from left; $f(x)$ has then an *ordinary* or *simple discontinuity* at x_0 . If the function has equal values of opposite sign at $x = x_0^+$ and $x = x_0^-$ (just to the right and just to the left of x_0), it possesses a *finite oscillatory discontinuity*. If the function grows beyond all limits as $x \rightarrow x_0$, either from right or left, it possesses an *infinite discontinuity* (either positive or negative); if the function, like $\tan x$ as $x \rightarrow \pi/2$, takes

infinite values of opposite sign to the right and to the left of x_0 , the function possesses there an *infinite oscillatory discontinuity*.

4. Convergence of a Sequence

An infinite sequence of numbers or functions

$$u_1, u_2, u_3 \cdot \cdot \cdot u_n, u_{n+1} \cdot \cdot \cdot$$

is convergent, and tends to a limit U , if u_n approaches U indefinitely closely as n increases beyond a large number. In mathematical terms

$$\lim_{n \rightarrow \infty} u_n = U, \quad \text{if } |U - u_n| < \epsilon \text{ for } n > N$$

Any sequence which is not convergent must either grow beyond all limits (positive or negative), and is then said to be *divergent*, or it *oscillates* either finitely or infinitely so that no value can be given.

5. Convergence of a Series

An infinite series is the sum of an infinite sequence of numbers or functions

$$S = u_1 + u_2 + u_3 + \cdot \cdot \cdot + u_n + u_{n+1} + \cdot \cdot \cdot$$

The series is *convergent* if the sequence of the partial sums S_N with N a large number is convergent. A series with positive and negative terms is *absolutely convergent* when the series of the absolute terms is convergent. If the convergence depends upon the presence of these positive and negative terms, the series is *conditionally convergent*; it may converge only for certain arrangements of the terms. If a series is not convergent, it may be either *divergent* or *oscillatory* (see "Convergence of a Sequence"). If the series is composed of functions of a variable x , it is said to be *uniformly convergent* in an interval a, b of the variable x , if it converges for all values of x in this interval.

6. Convergence of Infinite Integrals

Integrals of bounded and integrable functions, extending over infinite intervals of the variable, are called *infinite integrals*. An infinite integral is *convergent* when the limit of the integral exists as the interval of integration goes beyond all limits. In mathematical terms

$$\int_a^\infty f(x) dx = \lim_{M \rightarrow \infty} \int_a^M f(x) dx \quad \text{if this limit exists}$$

This definition applies also to the interval $-\infty, b$. Infinite integrals

are *absolutely convergent* if these limits exist for the absolute value of the integrand $|f(x)|$. If the integral extends over $-\infty, +\infty$ it is called *convergent* if the limit exists independent of whether we first let the upper interval go to infinity and then the lower, or vice versa. If the limit exists only by letting both intervals extend to infinity simultaneously, the integral is said to be *convergent, in the sense of Cauchy's principal value*; see Doetsch,^{D5} p. 91. Mathematically

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{M \rightarrow \infty} \int_{-M}^{+M} f(x) dx \quad (\text{Cauchy})$$

7. Convergence of Improper Integrals

Integrals over either finite or infinite intervals of the variable in which the integrand possesses one or a finite number of infinite discontinuities are frequently called *improper integrals*. If the infinite discontinuity occurs at $x = x_0 > a$, the improper integral with upper limit x_0 is *convergent* if its value exists no matter *how closely* we approach x_0 . In mathematical terms

$$\int_a^{x_0} f(x) dx = \lim_{\Delta \rightarrow 0} \int_a^{x_0 - \Delta} f(x) dx \quad \text{if this limit exists}$$

If the integral extends over an interval a, b with $b > x_0$, we need to subdivide this interval into the subintervals a, x_0 ; x_0, b . We then must demonstrate the convergence for the second subinterval at its lower limit, i.e.

$$\int_{x_0}^b f(x) dx = \lim_{\Delta \rightarrow 0} \int_{x_0 + \Delta}^b f(x) dx \quad \text{if this limit exists}$$

If the integral is convergent in both subintervals, it is convergent for the total interval a, b which might be infinite.

APPENDIX 3

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APPENDIX 4

MATRICES AND DETERMINANTS

In order to abbreviate the notation for systems of linear equations between two sets of unknown quantities, say

$$\begin{array}{ccccccc}
 a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & y_1 \\
 a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & y_2 \\
 \cdot & & \cdot \\
 \cdot & & \cdot \\
 \cdot & & \cdot \\
 a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = & y_n
 \end{array} \quad (1)$$

we can introduce the *matrices*

$$[a_{\alpha\beta}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad x_\beta = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \quad y_\alpha = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{bmatrix} \quad (2)$$

where $[a_{\alpha\beta}]$ is a *square matrix*, composed of the n rows and n columns of coefficients $a_{\alpha\beta}$ in (1), and where $x_\beta]$ and $y_\alpha]$ are *column matrices*,* composed of a single column. We define then the equivalent to (1) in matrix notation as

$$[a_{\alpha\beta}] \times x_\beta] = y_\alpha] \quad (3)$$

where the matrix product is so defined as to give (1), i.e., we multiply each term of the first row of $[a_{\alpha\beta}]$ with the correspondingly numbered x_β term of the matrix $x_\beta]$, and the sum of these n products equals y_α , the first row term of the matrix $y_\alpha]$; similarly for the second row of $[a_{\alpha\beta}]$, etc.

The *product of two more general matrices* is defined correspondingly. As an example take

* The symbol is taken from Guillemin; see reference 4 at end of this Appendix.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \times \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} \\ = \begin{bmatrix} a_1A_1 + b_1A_2 + c_1A_3; & a_1B_1 + b_1B_2 + c_1B_3; & a_1C_1 + b_1C_2 + c_1C_3 \\ a_2A_1 + b_2A_2 + c_2A_3; & a_2B_1 + b_2B_2 + c_2B_3; & a_2C_1 + b_2C_2 + c_2C_3 \\ a_3A_1 + b_3A_2 + c_3A_3; & a_3B_1 + b_3B_2 + c_3B_3; & a_3C_1 + b_3C_2 + c_3C_3 \end{bmatrix} \quad (4)$$

If now the first matrix had only two rows and three columns, we would enlarge it by adding a third row of zeros so as to make it a square matrix. The product with the second matrix is then found as before, but the right-hand matrix would have all zeros as the third row. If the second matrix were a single column matrix, we could also add second and third columns of zeros and thus get, on the right-hand side, a single column, again the first one. For the completion of a square matrix we shall always add zeros "behind" and "below" the given terms.

It must be obvious from the foregoing that matrix multiplication is *noncommutative*, i.e., we cannot interchange the two left-hand matrices in (4) without getting an entirely different result on the right-hand side. It is therefore necessary to differentiate carefully between *premultiplication* by a matrix and *postmultiplication*, the former indicating a matrix factor preceding the original matrix, the latter, conversely, a matrix factor succeeding the original matrix.

If we interchange rows and columns in a matrix, we obtain in general a new matrix, which we call the *transpose* of the original:

$$[a_{\alpha\beta}]_t = [a_{\beta\alpha}] = \begin{bmatrix} a_{11} & a_{21} & \cdot & \cdot & \cdot & a_{n1} \\ a_{12} & a_{22} & \cdot & \cdot & \cdot & a_{n2} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{1n} & a_{2n} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix} \quad (5)$$

where α, β refer to (2). The transposed matrix will be identical with the original, if the terms of the original are symmetrical about the main diagonal (defined by $a_{11} \rightarrow a_{nn}$). Such a matrix is actually called *symmetrical*. We have for all terms

$$a_{\alpha\beta} = a_{\beta\alpha} \quad (6)$$

This is the case for all passive linear network equations. The product of two symmetrical matrices is still not commutative, though we find that

$$[a_{\alpha\beta}] \times [b_{\alpha\beta}] = ([b_{\alpha\beta}] \times [a_{\alpha\beta}])_t \quad (7)$$

i.e., the two results are the *transpose* of each other.

A special symmetrical matrix is the *identity* or *unit matrix*

$$[U] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad (8)$$

If we multiply any matrix by this unit matrix, we obtain the original matrix identically.

The most important problem in connection with a *system of linear equations* (1) is its *solution* or, mathematically, its *inversion*, i.e., the expression of the unknown x_β values in terms of the known y_α values. A simple method of solution is by means of *Cramer's rule*, which writes x_β as the ratio of two *determinants*, namely

$$x_\beta = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1,\beta-1} & y_1 & a_{1,\beta+1} & \cdots & a_{1,n} \\ a_{21} & \cdots & a_{2,\beta-1} & y_2 & a_{2,\beta+1} & \cdots & a_{2,n} \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ a_{n1} & \cdots & a_{n,\beta-1} & y_n & a_{n,\beta+1} & \cdots & a_{n,n} \end{vmatrix}}{|a_{\alpha\beta}|} \quad (9)$$

The denominator is the complete *coefficient determinant* of (1) whereas the numerator is that same coefficient determinant with the β -column replaced by the column of the y_α values.

The determinant, though in appearance the same as a matrix, has a numerical value which is obtained by special computational rules. For example, the simplest (lowest order) determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad (10)$$

is the product of the *main diagonal* elements less the product of the "*conjugate*" *diagonal*. The value of the determinant of order three can be obtained by various methods. If we have

$$|a_{\alpha\beta}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad (11)$$

we call any lower order determinant formed from this one a *minor*; thus

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

is the minor obtained by canceling row 1 and column 2 in $|a_{\alpha\beta}|$. Generally $M_{\alpha\beta}$ is the minor of $|a_{\alpha\beta}|$ with row α and column β eliminated. The minor with the sign factor $(-1)^{\alpha+\beta}$ is called the *cofactor*

$$A_{\alpha\beta} = (-1)^{\alpha+\beta} M_{\alpha\beta} \quad (12)$$

The determinant (11) can be *expanded* along any row or column, and its value is given by

$$|a_{\alpha\beta}| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$

etc. Actually, this process can be applied at once to any higher order determinant so that it represents a systematic computational rule. If we call the value of the determinant Δ , then (9) can be expressed by expanding the numerator along the β -column, which is taken by the y_α values

$$x_\beta = \frac{1}{\Delta} [y_1A_{1\beta} + y_2A_{2\beta} + y_3A_{3\beta} + \cdots y_nA_{n\beta}] \quad (13)$$

The existence of this solution for any of the x_β depends primarily on the fact that Δ itself does not vanish. For any real physical system that we normally consider, Δ will be different from zero.

If then the solutions (13) exist, they can again be summarized in matrix form. We can write the column matrix x_β as the product of a square coefficient matrix and the column matrix y_α , where we keep the *original notation* from (2), *except that β now designates the rows and α the columns*. Thus

$$x_\beta = [b_{\beta\alpha}] \times y_\alpha \quad (14)$$

The coefficient comparison with (13) gives at once

$$b_{\beta\alpha} = \frac{A_{\alpha\beta}}{\Delta} \quad (15)$$

where the $A_{\alpha\beta}$ are the cofactors formed with respect to the coefficient determinant $|a_{\alpha\beta}|$ implied in (1) and identical in form to the matrix $[a_{\alpha\beta}]$ in (3).

Can we give a way to go directly from (1) or (3) to (14), i.e., give the *matrix solution* for the system of linear equations merely by matrix methods? If we compare (3) with (14), we see that we should pre-multiply on the left-hand side with a matrix whose product with $[a_{\alpha\beta}]$ results in the unit matrix (8). We define such a matrix as *inverse* to the original and define

$$[a_{\alpha\beta}]^{-1} \times [a_{\alpha\beta}] = [U] \quad (16)$$

To actually find the composition of the inverse matrix, we realize that premultiplication must occur on both sides of (3) so that the *final*

solution becomes

$$[a_{\alpha\beta}]^{-1} \times [a_{\alpha\beta}] \times x_{\beta} \equiv [U] \times x_{\beta} = [a_{\alpha\beta}]^{-1} \times y_{\alpha} \quad (17)$$

The last equation in (17) is identical with (14), and we have therefore the identification

$$[a_{\alpha\beta}]^{-1} = [b_{\beta\alpha}] = \left[\frac{A_{\alpha\beta}}{\Delta} \right] \quad (18)$$

where α is the running index, and β is the fixed index, so that

$$[a_{\alpha\beta}]^{-1} = \begin{bmatrix} \frac{A_{11}}{\Delta} & \frac{A_{21}}{\Delta} & \frac{A_{31}}{\Delta} & \dots & \frac{A_{n1}}{\Delta} \\ \frac{A_{12}}{\Delta} & & & \dots & \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \frac{A_{1n}}{\Delta} & \frac{A_{2n}}{\Delta} & & \dots & \frac{A_{nn}}{\Delta} \end{bmatrix} \quad (19)$$

We obtain the inverse to the square matrix $a_{\alpha\beta}$ by replacing each element $a_{\alpha\beta}$ by its cofactor $A_{\alpha\beta}$ divided by Δ , and then taking the transpose of the resulting matrix.

For practical purposes the finding of the inverse matrix as outlined in (19) and the direct application of Cramer's rule with determinants are identical processes, the matrix notation having the advantage of brevity. Actually, we can also find the inverse matrix by simpler manipulations, which require further study of matrix theory, as given particularly in Guillemin, listed in the references below.

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APPENDIX 5

ELEMENTS OF FUNCTIONS OF A COMPLEX VARIABLE

1. Concept of Analytic Functions

The complex variable $z = x + jy$ is in reality a special combination of the two independent real variables x and y which describe the location of any point in the x, y -plane or z -plane in Fig. A-5.1a,

$$z_1 = x_1 + jy_1 = |z_1|e^{j\varphi_1} = |z_1| \cos \varphi_1 + j|z_1| \sin \varphi_1 \quad (1)$$

where

$$|z_1| = r = (x^2 + y^2)^{1/2}, \quad \tan \varphi_1 = \frac{y_1}{x_1} \quad (2)$$

$|z_1|$ is the *modulus* or *absolute value*, and φ_1 is the *argument*.

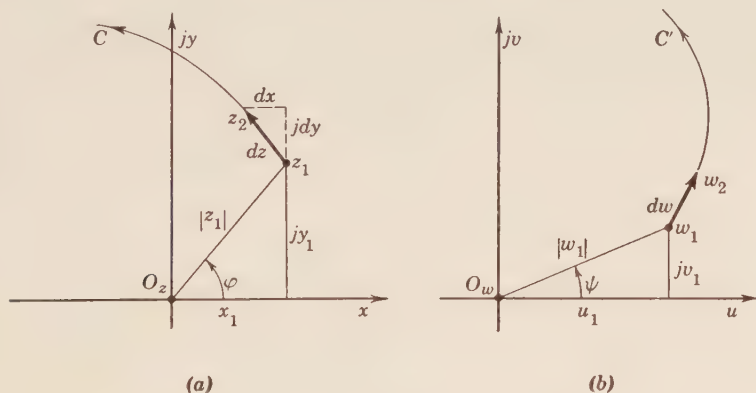


Fig. A-5.1. Concept of analytic function. (a) z -plane, (b) w -plane.

Any mathematical operation on the complex number z will again give a complex number (real numbers are included as special values); the realm of complex numbers is therefore *closed*, and we need not go beyond it at any time.

Any function of the complex variable z

$$w = f(z) = u(x, y) + jv(x, y) \quad (3)$$

will thus again be a complex number whose real and imaginary parts will be functions of the two real variables x and y . We can conceive of w as defining again a plane, as in Fig. A-5.1b, where at least one point w_1 will be found corresponding to point z_1 in the z -plane. Consider the simple function

$$w = f(z) = z^3 = r^3 e^{j3\varphi} = (x^3 - 3xy^2) + j(3x^2y - y^3) \quad (4)$$

We have

$$u(x, y) = x(x^2 - 3y^2), \quad v(x, y) = y(3x^2 - y^2)$$

bearing out that u and v are functions of both real variables x and y . In polar form we have

$$w = R e^{j\psi}, \quad R = r^3, \quad \psi = 3\varphi$$

and observe that for $0 < \varphi < (2\pi/3)$ we get $0 < \psi < 2\pi$, or one-third of the z -plane suffices to deliver all points of the w -plane. If we let φ go beyond $2\pi/3$, we cover the same w -points again and, indeed, a third time. Taking the inverse relation, $z = w^{1/3}$, we see at once that for every one point w_1 in the w -plane we find three distinct points in the z -plane, z_1' , z_1'' , and z_1''' , unless we were to specify three distinct layers of points in the w -plane, or three sheets, each one of these corresponding uniquely to one-third of the z -plane. We call the relationship $w = z^3$ a *single-valued function*, because one and only one value of w is obtained as we form z^3 . On the other hand, we call $z = w^{1/3}$ a *multiple-valued function*, because for one value of w more than one value of z can be found satisfying the functional relation.

The first task in studying a function $f(z)$ will then be to select a domain of z such that $w = f(z)$ is definitely *single-valued*. This requires restriction of z to one coverage of the z -plane, with a corresponding partial coverage of the w -plane. Any such selection we shall refer to in short as a single-valued function. It is, of course, irrelevant where we choose the exact boundaries of these corresponding branches. We shall frequently prefer either the positive real axis $x > 0$ as the starting boundary line so that $0 < \varphi < 2\pi$, or the negative real axis $x < 0$ so that $-\pi < \varphi < +\pi$, depending on convenience.

Next let us select a point z_2 close to z_1 and let it approach z_1 indefinitely. Because we are dealing with a plane, we can let this approach be made in an infinity of ways. If now the corresponding point w_2

always approaches w_1 indefinitely, we call $f(z)$ *continuous* at z_1 , and if in addition

$$\lim_{z_2 \rightarrow z_1} \frac{w_2 - w_1}{z_2 - z_1} \rightarrow \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz} = \frac{df(z)}{dz} \quad (5)$$

exists independent of the manner of approach, we call the function *differentiable* or *analytic* at z_1 . Because we must get the same result for approach along the real or the imaginary axes

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta w}{\Delta x} = \lim_{\Delta y \rightarrow 0} \frac{\Delta w}{j \Delta y} = \frac{dw}{dz}$$

we can deduce the partial derivative relations, using (3) for $w(z)$

$$\frac{\partial u}{\partial x} + j \frac{\partial v}{\partial x} = \frac{\partial u}{j \partial y} + j \frac{\partial v}{j \partial y}$$

This result leads to

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x} \quad (6)$$

the *Cauchy-Riemann conditions*, which can be shown to be both necessary and sufficient to assure differentiability of the function at point z . A single-valued function which satisfies the C-R conditions at z and in its immediate neighborhood is called *regular* or *analytic* at z . Where the C-R conditions are not satisfied, the function is called *singular*.

2. Classification of Analytic Functions and Singularities

Consider the function

$$w = \frac{1}{z - z_1}, \quad \frac{dw}{dz} = - \frac{1}{(z - z_1)^2} \quad (7)$$

Its derivative exists everywhere in the z -plane except at the point $z = z_1$. Actually, we can approach $z = z_1$ very closely and still find analyticity; $z = z_1$ is therefore called an *isolated singular point*. We notice also that the modified function

$$g(z) = (z - z_1)w = 1, \quad \frac{d}{dz} g(z) = 0$$

is analytic even at $z = z_1$; the singularity has been removed by multiplying w with the linear root factor $(z - z_1)$. We call the isolated singularity a removable singularity, or specifically a *pole of first order*.

Obviously, the function

$$w = \frac{1}{(z - z_1)^2}, \quad \frac{dw}{dz} = -\frac{2}{(z - z_1)^3}$$

has a *pole of second order* at $z = z_1$, which can be removed by multiplying with the square of the root factor $(z - z_1)$.

If we define as *rational functions* any polynomial or fraction of two polynomials in z

$$w = f(z) = \frac{N(z)}{D(z)} = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0} \quad (8)$$

we realize at once that such functions can have only a finite number of poles of *finite order*. Indeed, using the partial fraction expansion from algebra, we can set these poles into evidence. We need to examine also the behavior of the function as $z \rightarrow \infty$. Let us first go back to the simple polynomial

$$w = f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \quad (9)$$

There are no singularities in the finite plane. To study the point $z = \infty$, we introduce a transformation $z = 1/t$ and study the new function of t at the origin, namely, from (9)

$$w \left(\frac{1}{t} \right) = f \left(\frac{1}{t} \right) = \frac{1}{t^n} (a_n + a_{n-1} t + a_{n-2} t^2 + \cdots + a_0 t^n) = \frac{h(t)}{t^n}$$

The derivative

$$\frac{dw}{dt} = \frac{1}{t^{2n}} \left(t^n \frac{d}{dt} h(t) - n t^{n-1} h(t) \right)$$

does not exist at $t = 0$, so $t = 0$ is a pole of n th order because $t^n w(t)$ is regular at $t = 0$. We therefore conclude that $w = f(z)$ from (9) has a pole of n th order at $z = \infty$. Returning to (8), we see that

$$\lim_{z \rightarrow \infty} f(z) = \frac{a_n}{b_m} z^{n-m} \quad (10)$$

At $z = \infty$, the function will have a pole of order $(n - m)$ if $n > m$, and it will be regular if $n \leq m$.

The *irrational function*

$$w = (z)^{1/2}, \quad \frac{dw}{dz} = \frac{1}{2(z)^{1/2}} \quad (11)$$

will be singular at $z = 0$. However, because of the square root, to one

value of z we can choose two values of w . If we express z in terms of w , $z = w^2$, we see that we must expect this double valuedness if w covers the entire plane, since this is analogous to the function defined in (4). We call here $z = 0$ a *branch point* and introduce a *branch cut* shown in Fig. A-5.2, a *barrier* around the branch cut which is imaged in the w -plane parallel to the imaginary axis, defining the right-hand

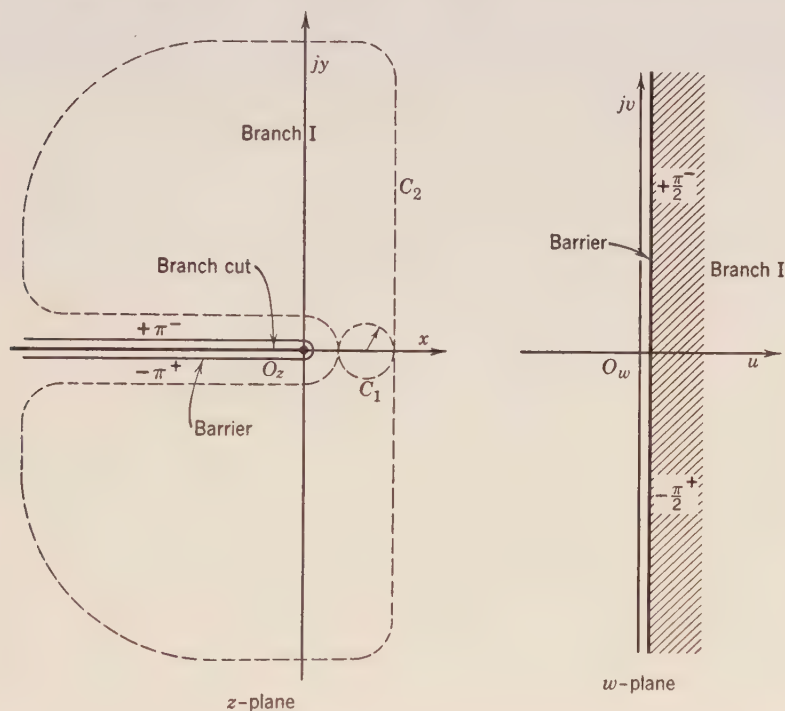


Fig. A-5.2. Branch cut and barrier in z - and w -planes for $w = (z)^{1/2}$.

half plane as branch I and the left-hand half plane as branch II. Figure A-5.3 shows the Riemann z -surface in perspective view for better visualization.

The general *algebraic* function is defined by the polynomial

$$P(z, w) = 0 \quad (12)$$

containing both z and w to any finite power. It will possess finite numbers of poles and branch points.

Another elementary function is the *transcendental*

$$w = f(z) = e^z, \quad \frac{dw}{dz} = e^z$$

which is regular in the entire finite z -plane. At $z = \infty$, it must be singular because the derivative does not exist there. To examine the type of singularity we might best use the power series which, in fact, serves as definition, namely

$$w = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots \quad (13)$$

Comparing with (9), we infer that e^z must have an isolated singularity

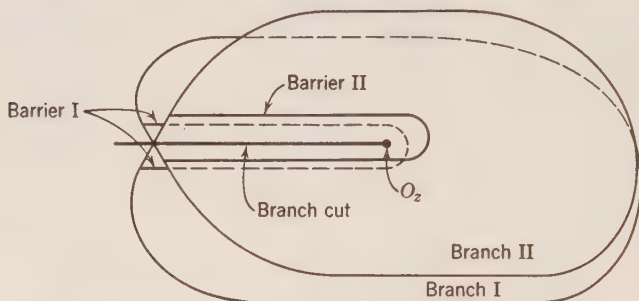


Fig. A-5.3. Riemann z -surface for $w = (z)^{1/2}$.

of infinite order at $z = \infty$, irremovable by any finite power of z . We define this as *essential singularity*. The inverse function

$$z = \ln w, \quad \frac{dz}{dw} = \frac{1}{w}$$

has at $w = 0$ a branch point or winding point of infinite order, because the logarithmic function has an infinite number of sheets. We usually define its *principal value* as

$$z = \ln (|w|e^{j\theta}) = \ln |w| + j\theta, \quad -\pi < \theta < \pi \quad (14)$$

As a final example we take the transcendental function

$$w = f(z) = \tan z, \quad \frac{dw}{dz} = \frac{1}{\cos^2 z} \quad (15)$$

This function possesses simple poles (of first order) at an infinite number of isolated points

$$z = \pm \frac{\pi}{2}, \quad \pm \frac{3\pi}{2}, \quad \cdots, \quad \pm (2n-1) \frac{\pi}{2}, \quad \cdots$$

as we can verify by representing $\cos z$ in product form in accordance

with Weierstrass' development

$$\cos z = \left[1 - \left(\frac{2z}{\pi} \right)^2 \right] \left[1 - \left(\frac{2z}{3\pi} \right)^2 \right] \left[1 - \left(\frac{2z}{5\pi} \right)^2 \right] \cdots \left[1 - \left(\frac{2z}{(2n-1)\pi} \right)^2 \right] \cdots \quad (16)$$

Functions which have only poles in the finite plane are generally called *meromorphic functions*. Though the number of poles of $\tan z$ is infinite, they are *denumerable* or *countable*, because of their exactly known periodic spacing. At infinity, $\tan z$ possesses an essential singularity.

3. Integration

For single-valued analytic functions, the integral over any closed path vanishes as long as the path does not include or go through any singularity. This is obvious for the simple integral

$$\oint z^3 dz = \frac{1}{4} z^4 \Big|_{\mathcal{Q}} = 0$$

because the integral function z^4 is single-valued in the entire z -plane and therefore will come back to the starting value for any closed path. This will also be true for the integral of any single-valued analytic function because the existence of its derivative, which is the original function, requires it to be analytic! This is the *fundamental integral theorem of Cauchy, or of Cauchy-Goursat*.^{*} It permits the deformation of any path into any arbitrary shape as long as in so doing we are not passing into or over a singular point of $f(z)$. Thus, in Fig. A-5.2, we can let the small circle C_1 become the very large closed path C_2 , following partly the barrier curve, or, conversely, let C_2 shrink into C_1 .

The integral over any circle with center at the origin

$$\oint \frac{dz}{z} = \ln z \Big|_{\mathcal{Q}} = \ln (|z| e^{j\varphi})_{\varphi=0}^{\varphi=2\pi} = j2\pi \quad (17)$$

does not vanish, because $\ln z$ has a branch point at the origin. The function cannot return to its starting value, so that we need to place a branch cut, e.g., at $\varphi = 0$ in Fig. A-5.4a. If we were to choose a closed path not enclosing the origin, such as C_2 in Fig. A-5.4a, the result of the integration would again be zero. Even the integral along the closed path over $C'C''$ in Fig. A-5.4b would vanish because we can reconstruct it, transforming the multiply-connected annular region into a singly-connected region in accordance with Fig. A-5.4c.

^{*} See reference 1, p. 81, at end of this Appendix.

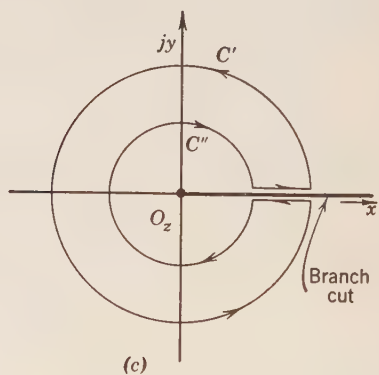
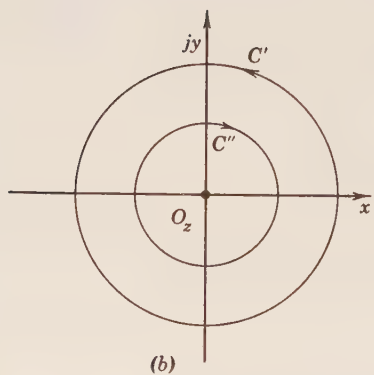
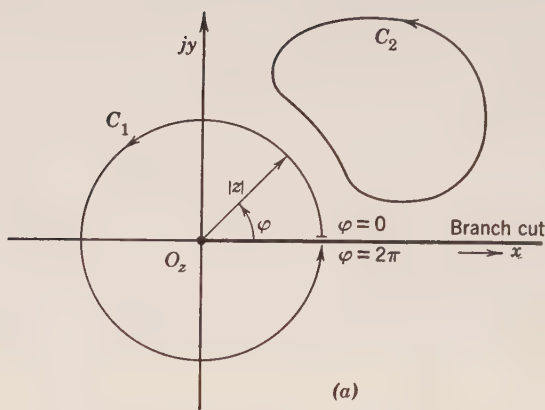


Fig. A-5.4. Integration of z^{-1} over a closed path.

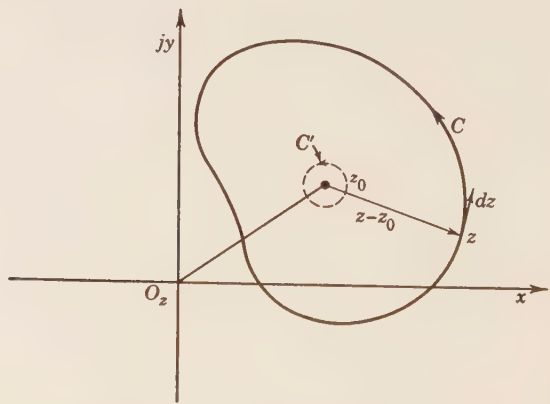


Fig. A-5.5. Cauchy's integral formula.

If the function $f(z)$ is single-valued and analytic within and on a closed path C , the integral

$$\oint \frac{f(z)}{z - z_0} dz = 2\pi j f(z_0) \quad (18)$$

where z_0 is any point within C , as shown in Fig. A-5.5. We can let path C shrink to a very small circle C' around z_0 . Let us introduce $z - z_0 = \zeta$, and, remembering that $f(z) = f(z_0 + \zeta)$ must be continuous near z_0 , we can expand into the Taylor series

$$f(z_0 + \zeta) = f(z_0) + \zeta f'(z_0) + \cdots$$

Thus (18) becomes

$$f(z_0) \oint \frac{d\zeta}{\zeta} + f'(z_0) \oint d\zeta + \cdots$$

With (17), the first term gives (18) directly; all other terms vanish because of the regularity of the integrand, and actually need not be considered as $\zeta \rightarrow 0$. This *Cauchy fundamental integral formula* (18) permits the evaluation of the single-valued and analytic function $f(z)$ at $z = z_0$ by integration over a closed path along which $f(z)$ is known. Moreover, it defines the existence of all higher order derivatives of this function, because we can differentiate (18) with respect to z_0 and obtain

$$\frac{d}{dz_0} f(z_0) = \frac{1}{2\pi j} \oint \frac{f(z)}{(z - z_0)^2} dz \quad (19)$$

and

$$\frac{d^{n-1}}{dz_0^{n-1}} f(z_0) = \frac{(n-1)!}{2\pi j} \oint \frac{f(z)}{(z - z_0)^n} dz \quad (20)$$

In turn, the existence of all derivatives of an analytic function assures that they are also analytic functions (satisfying the C-R conditions). We can therefore expand any analytic function at a point z_0 into a convergent Taylor series

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \cdots \quad (21)$$

The convergence is assured for the range of analyticity of $f(z)$, because the derivatives are analytic in the same range. Series (21) can be differentiated and integrated term by term.

4. Residues

Rational functions which can possess only poles as singularities are the most important type of functions for network analysis. If a func-

tion $f(z)$ has a pole of first order at $z = z_0$, we must be able to write it in the form

$$f(z) = \frac{g(z)}{z - z_0}$$

because of the removable character of the pole. The numerator function $g(z)$ is analytic everywhere, even at $z = z_0$. The closed integral over C encircling z_0 as in Fig. A-5.5 can now be evaluated easily. We have by (18)

$$\oint f(z) dz = \oint \frac{g(z)}{z - z_0} dz = 2\pi j g(z_0) = 2\pi j [(z - z_0)f(z)]_{z=z_0} \quad (22)$$

where $g(z_0)$ is called the *residue of the function $f(z)$ at the simple pole $z = z_0$* . If we expand $g(z)$ into the Taylor series (21), we obtain for $f(z)$

$$f(z) = \frac{g(z_0)}{z - z_0} + g'(z_0) + \frac{z - z_0}{2!} g''(z_0) + \cdots \quad (23)$$

a series development which sets the pole into evidence. This is called the *Laurent series* and is valid between the small circle C' surrounding the pole z_0 and the convergence range of the Taylor series for $g(z)$.

For a *general rational function* as in (8), which possesses only simple poles, we can expand about each pole into a Laurent series (23) valid up to the nearest neighboring pole. The integral over a large closed path C enclosing all the poles is then simply the *sum of the residues* for all the simple poles. This integral is evaluated best by breaking the rational function (8) into partial linear fractions, namely

$$f(z) = \frac{N(z)}{D(z)} = \sum_{\alpha=1}^n \left(\frac{N(z)}{dD(z)/dz} \right)_{z=z_\alpha} \frac{1}{z - z_\alpha} \quad (24)$$

if we assume that $D(z)$ is a polynomial of degree n and $N(z)$ of any lesser degree. The partial fraction expansion is a purely algebraic matter and can be taken from any book on linear algebra. The result of integration around all the poles is then

$$\oint f(z) dz = 2\pi j \sum_{\alpha=1}^n \left(\frac{N(z)}{dD(z)/dz} \right)_{z=z_\alpha} \quad (25)$$

For a *pole of higher order*, because of the ability to remove it, we must be able to write

$$f(z) = \frac{h(z)}{(z - z_0)^n}$$

where $h(z)$ is entirely analytic even at $z = z_0$. The integral around it will give as residue by means of (20)

$$\oint f(z) dz = \oint \frac{h(z)}{(z - z_0)^n} dz = \frac{2\pi j}{(n-1)!} \left(\frac{d^{n-1}}{dz^{n-1}} h(z) \right)_{z=z_0} \quad (26)$$

where we can replace $h(z) = f(z)(z - z_0)^n$. If we expand $h(z)$ in the Taylor series (21) and introduce it into $f(z)$, we get the corresponding Laurent series expansion about the n th order pole

$$\begin{aligned} f(z) = \frac{h(z_0)}{(z - z_0)^n} + \frac{h'(z_0)}{(z - z_0)^{n-1}} + \frac{h''(z_0)}{2!(z - z_0)^{n-2}} \\ + \cdots + \frac{h^{(n-1)}(z_0)}{(n-1)!(z - z_0)} + \frac{h^n(z_0)}{n!} \\ + \frac{h^{n+1}(z_0)}{(n+1)!} (z - z_0) + \cdots \quad (27) \end{aligned}$$

By extrapolation, we can deduce that the Laurent series at a pole of infinite order will possess an infinite number of negative power terms, which is characteristic for this essential singularity.

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